CS 70 Fall 2003	Discrete Mathematics for CS Wagner	HW 2

Due Thursday September 11

Formatting your solution: Please put at the top of your solution the following information: your username on `cory.eecs`, your full name, the string "CS70, Fall 2003, HW #2", your section number, and your partners.

1. **(5 pts.) Any questions?**
   What's the one thing you'd most like to see explained better in lecture or discussion sections? A one-line answer would be appreciated.

   (Sometimes we botch the description of some concept, leaving people confused. Sometimes we omit things people would like to hear about. Sometimes the book is very confusing on some point. Here's your chance to tell us what those things were. All feedback is welcome.)

2. **(20 pts.) Simple induction**
   Prove that $2^n < n!$ for all integers $n \geq 4$.

3. **(20 pts.) Simple induction**
   Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for all integers $n \in \mathbf{N}$.

4. **(15 pts.) A pizza proof**
   Working at the local pizza parlor, I have a stack of unbaked pizza doughs. For a most pleasing presentation, I wish to arrange them in order of size, with the largest pizza on the bottom. I know how to place my spatula under one of the pizzas and flip over the whole stack above the spatula (reversing their order). This is the only move I know that can change the order of the stack; however, I am willing to keep repeating this move until I get the stack in order. Is it always possible to get the pizzas in order? Prove your answer.

5. **(15 pts.) Strong induction**
   Chocolate often comes in rectangular bars marked off into smaller squares. It is easy to break a larger rectangle into two smaller rectangles along any of the horizontal or vertical lines between the squares. Suppose I have a bar containing k squares and wish to break it down into its individual squares. Prove that *no matter which way I break it*, it will take exactly $k-1$ breaks to do this.

6. **(25 pts.) You be the grader**
   Assign a grade of A (correct) or F (failure) to each of the following proofs. If you give a F, please explain exactly what is wrong with the structure or the reasoning in the *proof*. You should justify all your answers (remember, saying that the claim is false is *not* a justification).

   (a) **Theorem 0.1**: *For all positive integers n,* $\sum_{i=1}^{n} i = (n-1)(n+2)/2$.
   Proof: The proof will be by induction.
   Base case: The claim is valid for $n = 1$.
   Induction step: Assume that $\sum_{i=1}^{k} i = (k-1)(k+2)/2$. Then $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$. By

the inductive hypothesis, $\sum_{i=1}^{k+1} i = (k-1)(k+2)/2+(k+1)$. Collecting terms and simplifying, the right-hand side becomes $k(k+3)/2$. Thus $\sum_{i=1}^{k+1} i = ((k+1)-1)((k+1)+2)/2$, which completes the induction step. $\square$

(b) **Theorem 0.2**: *For every $n \in \mathbf{N}$, n^2+n is odd.*

Proof: The proof will be by induction.
Base case: The natural number 1 is odd.
Inductive step: Suppose $k \in \mathbf{N}$ and k^2+k is odd. Then,

$$(k+1)^2+(k+1) = k^2+2k+1+k+1 = (k^2+k)+(2k+2)$$

is the sum of an odd and an even integer. Therefore, $(k+1)^2+(k+1)$ is odd. By the Principle of Mathematical Induction, the property that n^2+n is odd is true for all natural numbers n. $\square$

(c) **Theorem 0.3**: *For all $x,n \in \mathbf{N}$, if $nx = 0$ and $n > 0$, then $x = 0$.*

Proof: The proof will be by induction.
Base case: If $n = 1$, then the equation $nx = 0$ implies $x = 0$, since $nx = 1 \cdot x = x$ in this case.
Induction step: Fix $k > 0$, and assume that $kx = 0$ implies $x = 0$. Suppose that $(k+1)x = 0$. Note that $(k+1)x = kx+x$, hence we can conclude that $kx+x = 0$, or in other words, $kx = -x$. Now there are two cases:

Case 1: $x = 0$. In this case, $kx = -x = -0 = 0$, so $kx = 0$. Consequently, the inductive hypothesis tells us that $x = 0$.

Case 2: $x > 0$. In this case, $-x < 0$ (since $x > 0$). At the same time, $kx \geq 0$ (since $k,x \geq 0$). But this is impossible, since we know $kx = -x$. We have a contradiction, and therefore Case 2 cannot happen.

In either case, we can conclude that $x = 0$. This completes the proof of the induction step. $\square$

(d) **Theorem 0.4**: *For all $x,y,n \in \mathbf{N}$, if $\max(x,y) = n$, then $x = y$.*

Proof: The proof will be by induction.
Base case: Suppose that $n = 0$. If $\max(x,y) = 0$ and $x,y \in N$, then $x = 0$ and $y = 0$, hence $x = y$.
Induction step: Assume that, whenever we have $\max(x,y) = k$, then $x = y$ must follow. Next suppose x,y are such that $\max(x,y) = k+1$. Then it follows that $\max(x-1,y-1) = k$, so by the inductive hypothesis, $x-1 = y-1$. In this case, we have $x = y$, completing the induction step. $\square$

(e) **Theorem 0.5**: $\forall n \in \mathbf{N}.\ n^2 \leq n.$

Proof: The proof will be by induction.
Base case: When $n = 0$, the statement is $0^2 \leq 0$ which is true.
Induction step: Now suppose that $k \in \mathbf{N}$, and $k^2 \leq k$. We need to show that

$$(k+1)^2 \leq k+1$$

Working backwards we see that:

$$\begin{aligned} (k+1)^2 &\leq k+1 \\ k^2+2k+1 &\leq k+1 \\ k^2+2k &\leq k \\ k^2 &\leq k \end{aligned}$$

So we get back to our original hypothesis which is assumed to be true. Hence, for every $n \in \mathbf{N}$ we know that $n^2 \leq n$. $\square$