Implementation of Chua's Circuit with a Cubic Nonlinearity

Guo-Qun Zhong

Abstract—**This paper reports an implementation of Chua's circuit with a smooth nonlinearity, described by a cubic polynomial. Some bifurcation phenomena and chaotic attractors observed experimentally from the laboratory model and simulated by computer for the model are also presented. Comparing both the observations and simulations, the results are satisfactory.**

I. Introduction

The well-known Chua's circuit shown in Fig. 1, in which the nonlinearity of the Chua's diode is described by a piecewise-linear function, has been studied worldwide since it was invented by Chua in 1983 and confirmed by computer simulation and experimental observation, respectively, [1]–[4].

The state equations describing the circuit are as follows:

$$\left.\begin{aligned} \frac{dv_{c1}}{dt} &= \frac{1}{C_1}\left[\frac{1}{R}(v_{c2}-v_{c1}) - g(v_{c1})\right] \\ \frac{dv_{c2}}{dt} &= \frac{1}{C_2}\left[\frac{1}{R}(v_{c1}-v_{c2}) + i_L\right] \\ \frac{di_L}{dt} &= \frac{1}{L}\left[-v_{c2} - R_0 i_L\right] \end{aligned}\right\} \quad (1)$$

where $g(v_R)$ is a piecewise-linear function defined by

$$g(v_R) = G_b v_R + \frac{1}{2}(G_a - G_b)[|v_R + E| - |v_R - E|] \quad (2)$$

and R_0 denotes the small positive resistance of the inductor[1]. Most interesting chaotic phenomena and chaotic dynamics can be described by this *piecewise-linear* Chua's equation.

Recent numerical simulations reveal, however, that not all features of a real circuit are captured correctly by this piecewise-linear circuit [6]. It is therefore desirable to realize a *smooth* nonlinearity described by the following cubic polynomial for Chua's circuit:

$$g(v_R) = a_0 + a v_R + b v_R^2 + c v_R^3 \quad (3)$$

In Section II we present a practical implementation of this cubic nonlinearity. Some bifurcation sequences and chaotic attractors observed experimentally and simulated via the software INSITE are presented in Section III.

II. Practical Implementation of a Cubic Polynomial

The basic circuit we use to realize the cubic polynomial (3) is a multiplier circuit with a feedback loop, as shown in Fig. 2(a). The equivalent circuit of the multiplier circuit is shown in Fig. 2(b). By

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The author is with the Electronics Research Laboratory and the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720 on leave from Guangzhou Institute of Electronic Technology, Academia Sinica, Guangzhou 510070, People's Republic of China.

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[1] In the recent global unfolding of Chua's circuit [5], R_0 may assume any positive or negative value. This generalization is now called Chua's oscillator [2].

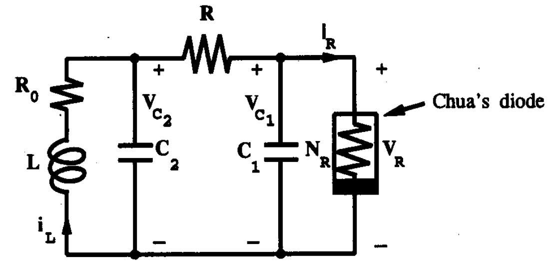


Fig. 1. Chua's circuit.

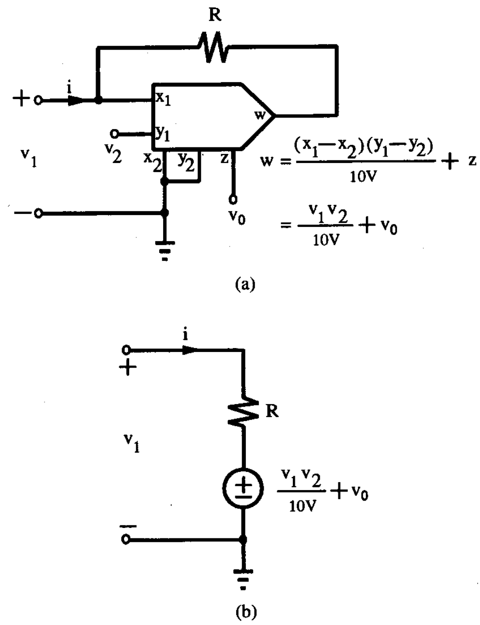


Fig. 2. (a) The multiplier circuit with a feedback loop. (b) The equivalent circuit of the multiplier circuit of (a).

applying Kirchhoff's Voltage Law to the equivalent circuit, we have

$$i = [v_1 - \frac{v_1 v_2}{10V} - v_0]\frac{1}{R} \quad (4)$$

where the factor $10V$ is an inherent scaling voltage in the multiplier, and v_0 is a dc voltage. Obviously, when $v_2 = v_1$ and $v_2 = v_1^2$, we obtain

$$i_1 = [v_1 - \frac{v_1^2}{10V} - v_0]\frac{1}{R} \quad (5)$$

and

$$i_2 = [v_1 - \frac{v_1^3}{10V} - v_0]\frac{1}{R} \quad (6)$$

respectively.

By adding (5) and (6), we obtain the following desired cubic polynomial:

$$i = a_0 + a v_1 + b v_1^2 + c v_1^3 \quad (7)$$

where $i = i_1 + i_2, a_0 = -\frac{2v_0}{R}, a = \frac{2}{R}, b = -\frac{1}{R}\frac{1}{10V}, c = -\frac{1}{R}\frac{1}{10V}$.

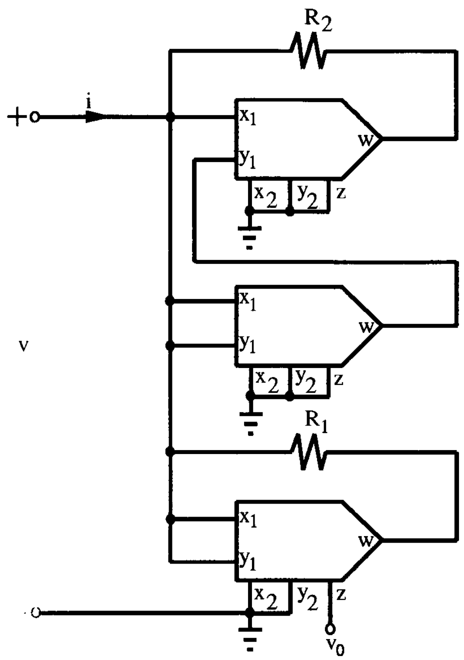
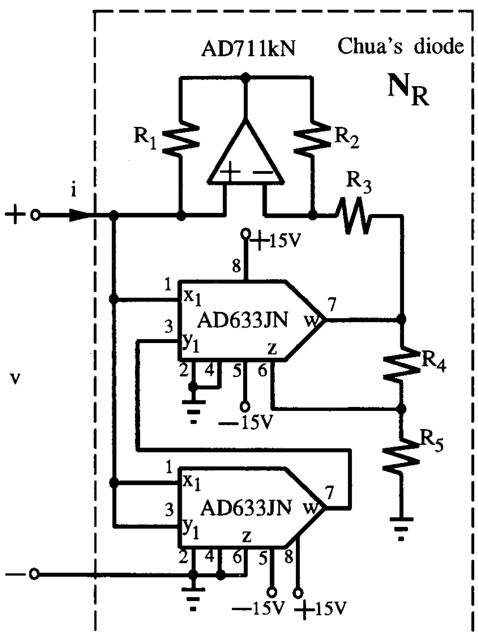
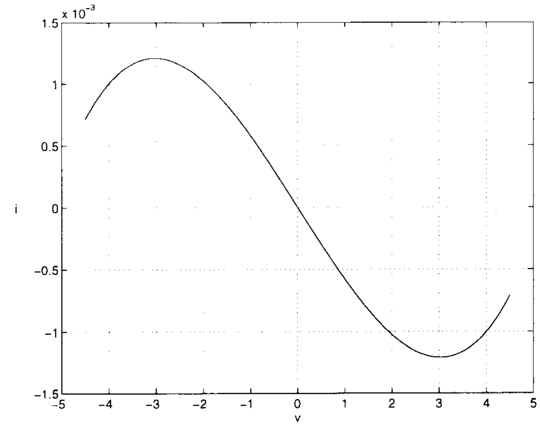


Fig. 3. A circuit implementation for a cubic-polynomial Chua's diode.



(a)



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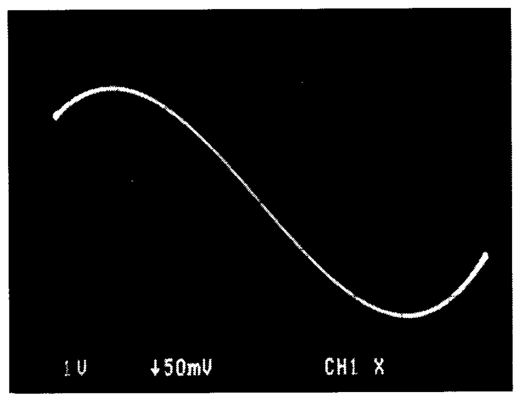


(c)

Fig. 4. (a) Practical circuit for realizing a cubic polynomial $v - i$ characteristic. (b) The calculated $v - i$ characteristic of Chua's diode N_R with a cubic nonlinearity $i = av + cv^3$, where $a = -0.599mS$, $c = 0.0218mS/V^2$. Horizontal axis v, scale: $1V$/div. Vertical axis i, scale: $0.5mA$/div. (c) The measured $v - i$ characteristic of Chua's diode N_R with a cubic nonlinearity $i = av + cv^3$. Horizontal axis v, scale: $1V$/div. Vertical axis i, scale: $0.5mA$/div, $a = -0.59mS, c = 0.02mS/V^2$.

Fig. 4. *Cont.*

The circuit implementation for the cubic polynomial (7) is shown in Fig. 3.

Note from the procedure above that any polynomial with higher-order terms and real coefficients can be realized in the same way. By choosing the signs of the resistors R_1 and R_2, the signs of the coefficients a_0, a, b, and c can be changed, respectively.

III. Bifurcation and Chaos in Chua's Circuit with a Cubic Nonlinearity

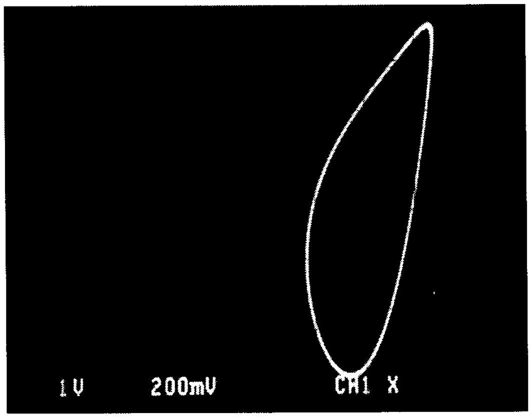
1. Practical Implementation of Chua's Circuit with a Cubic Nonlinearity

Since the desired $v - i$ characteristic of the nonlinear resistor N_R in Chua's circuit is an odd-symmetric function with respect to origin, here we use the cubic polynomial (7) with the coefficients $a_0 = 0, a < 0, b = 0$, and $c > 0$ for the nonlinearity of Chua's circuit in Fig. 1, i.e.,

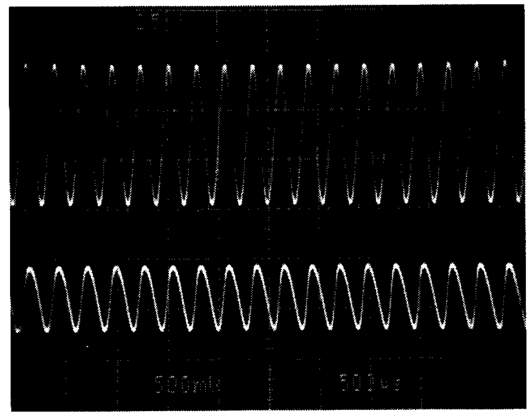
$$i_R = g(v_R) = av_R + cv_R^3 \tag{8}$$

where $a < 0$ and $c > 0$.

The practical circuit for realizing the cubic polynomial (8) is shown in Fig. 4(a). The two-terminal nonlinear resistor N_R consists of one



(a)



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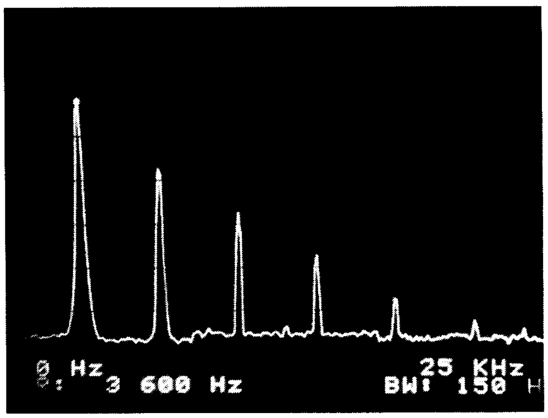

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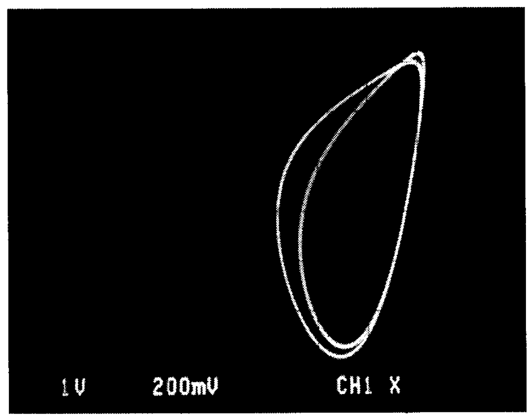

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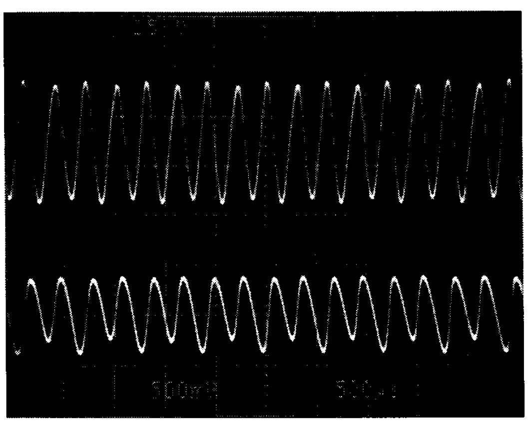

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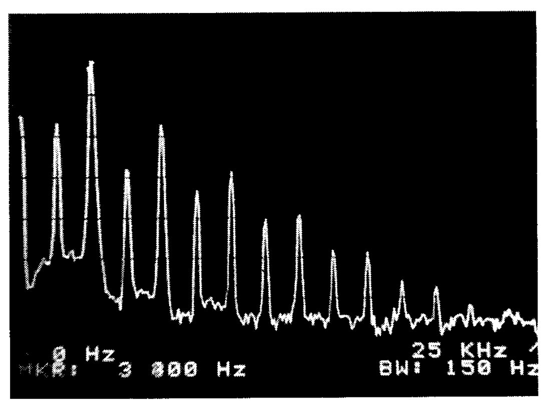
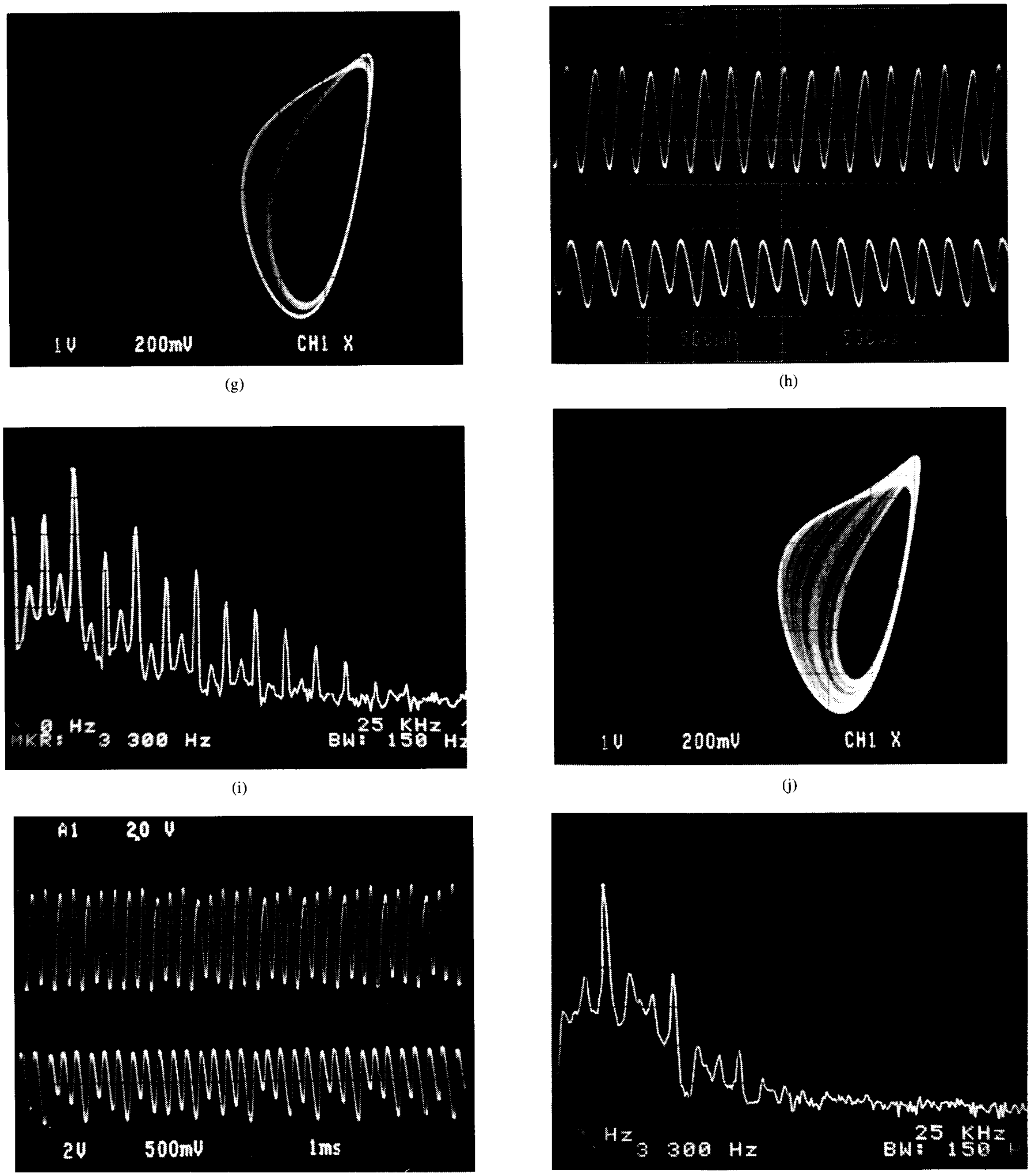


(f)

Fig. 5. Bifurcation sequence with respect to the parameter R. (a), (d), (g), (j), (m), and (p) Phase portraits in the $v_{c1} - v_{c2}$ plane. Horizontal axis v_{c1}, scale: $1V$/div. Vertical axis v_{c2}, scale: $0.2V$/div. (b), (e), (h), (k), (n), and (q) Time waveforms. Horizontal axis t, scale: (b), (e), and (h) $500\mu S$/div, (k), (n), and (q) $1mS$/div. Vertical axis v_{c2} (top) and v_{c1} (bottom), scale: $0.5V$/div for v_{c2}, $2V$/div for v_{c1}. (c), (f), (i), (l), (o), and (r) Spectra of voltages v_{c1}, scale: $10db$/div. Parameter values: $C_1 = 7nF, C_2 = 78nF, L = 18.91mH, R_0 = 14.99\Omega$ (the internal resistance of the inductor L), $a = -0.59mS, c = 0.02mS/V^2$. (a)–(c) $R = 2200\Omega$, (a) period-1 limit cycle. (d)–(f) $R = 2103\Omega$, (d) period-2 limit cycle. (g)–(i) $R = 2090\Omega$, (g) period-4 limit cycle. (j)–(l) $R = 2083\Omega$, (j) intermittency of type I. (m)–(o) $R = 2033\Omega$, (m) spiral Chua's attractor. (p)–(r) $R = 1964\Omega$, (p) Double-Scroll Chua's attractor.

Op Amp, two multipliers and five resistors. In the circuit, we utilize two analog multipliers AD633JN and an Op Amp AD711kN, both manufactured by Analog Devices, Inc.. The connections of the Op Amp AD711kN and the resistors R_1, R_2, and R_3 form an equivalent negative resistance R_e since we have $R_e = -R_3$ when $R_1 = R_2$ and the Op Amp operates in its linear region, in order to obtain the



(g) (h) (i) (j) (k) (l)

Fig. 5. *Cont.*

desired coefficients $a < 0$ and $c > 0$ in (8). Inversely, in the case where R_ϵ is a positive resistance, we will obtain $a > 0$ and $c < 0$ in (8). The driving-point $v - i$ characteristic of N_R is as below:

$$i_R = g(v_R) = -\frac{1}{R_3}v_R + \frac{R_4 + R_5}{R_3 R_4}\frac{1}{10V}\frac{1}{10V}v_R^3 = av_R + cv_R^3 \tag{9}$$

where $a = -\frac{1}{R_3}$, $c = \frac{R_4+R_5}{R_3R_4}\frac{1}{10V}\frac{1}{10V}$. The factor $10V$ is an inherent scaling voltage in the multiplier, as mentioned above. The network connected by the resistors R_4 and R_5 increases the gain of the system by the ratio $\frac{R_4+R_5}{R_4}$ in order to obtain a variable scale factor $\frac{R_4+R_5}{R_4}$. This ratio is limited to 100 in practical applications[2]. Usually, choose $R_4 \geq 1k\Omega$, and $R_5 \leq 100k\Omega$. Note that the coefficients a and c can

[2]Refer to *Data Converter Reference Manual* by Analog Devices.

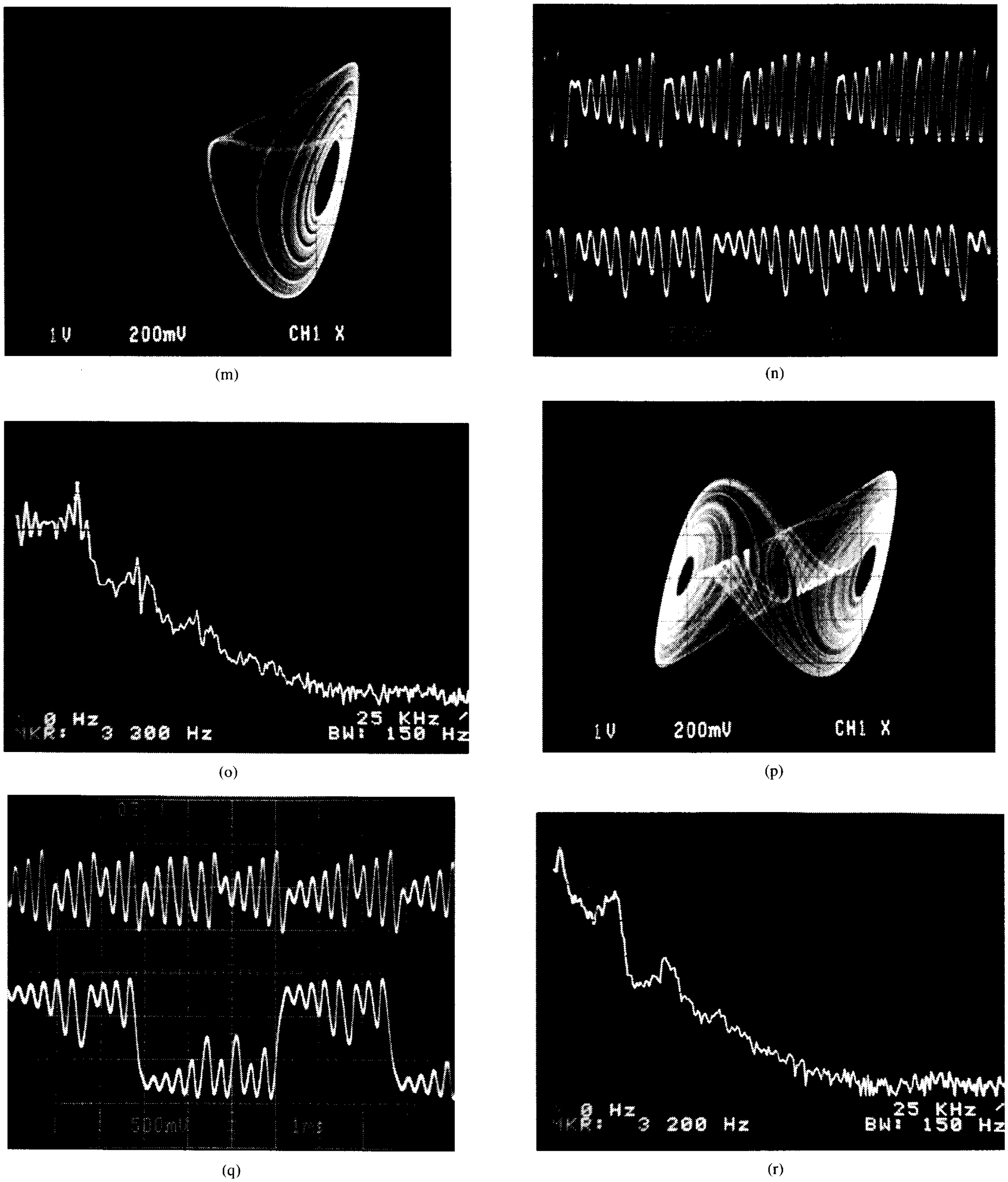


Fig. 5. *Cont.*

be adjusted by tuning the resistance R_3, and c can independently be adjusted by tuning the resistance R_5.

In our experimental model, we choose $R_1 = R_2 = 2k\Omega, R_3 = 1.668k\Omega, R_4 = 3.01k\Omega$, and $R_5 = 7.91k\Omega$. The $v - i$ characteristics of the Chua's diode N_R calculated according to the polynomial (9) and measured experimentally, based on the parameter values listed above, are shown in Figs. 4(b) and 4(c), respectively, where $a = -0.599mS, c = 0.0218mS/V^2$. Note that there is a very good agreement between the two curves.

2. Bifurcation and Chaos from Chua's Circuit with a Cubic Nonlinearity

The state equations for Chua's circuit in Fig. 1 with a cubic nonlinearity are as follows:

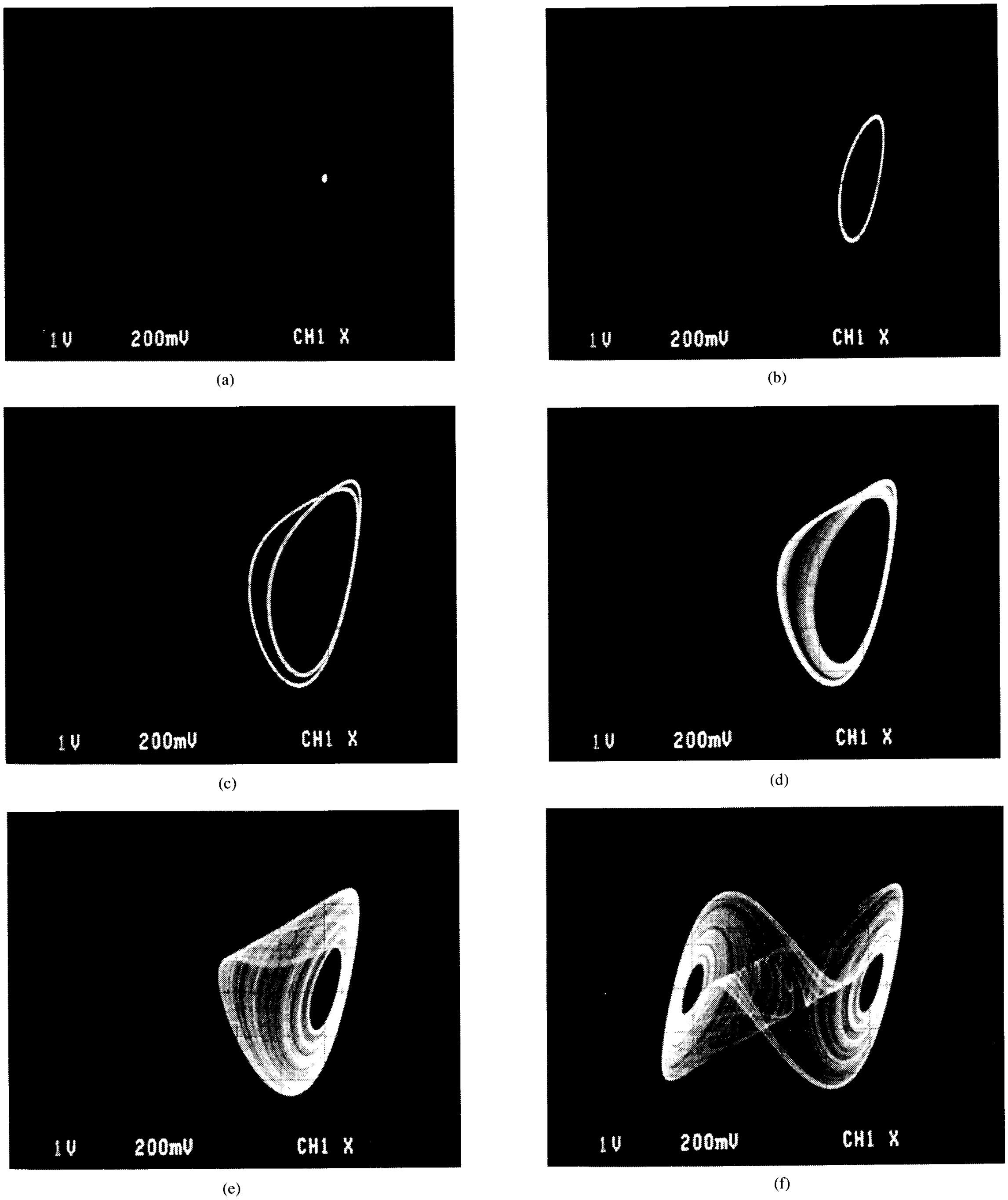


Fig. 6. Bifurcation sequence with respect to the parameter C_2. (a)–(g) Phase portraits in $v_{c1} - v_{c2}$ plane. Horizontal axis v_{c1}, scale: 1V/div. Vertical axis v_{c2}, scale: 0.2V/div. (h) Spectrum of voltage v_{c1}, scale: $10db$/div. Parameter values: $C_1 = 7nF$, $L = 18.91mH$, $R = 1964\Omega, R_0 = 14.99\Omega$ (the internal resistance of the inductor L), $a = -0.59mS, c = 0.02mS/V^2$. (a) $C_2 = 30nF$, dc equilibrium point. (b) $C_2 = 32nF$, period-1 limit cycle. (c) $C_2 = 54nF$, period-2 limit cycle. (d) $C_2 = 57nF$, intermittency of type I. (e) $C_2 = 64nF$, spiral Chua's attractor. (f) $C_2 = 78nF$, Double-Scroll Chua's attractor. (g) $C_2 = 600nF$, Double-Scroll Chua's attractor having a much lower frequency spectrum. (h) $C_2 = 600nF$, spectrum of voltage v_{c1}.

$$\left.\begin{aligned} \frac{dv_{c1}}{dt} &= \frac{1}{C_1}\left[\frac{1}{R}(v_{c2} - v_{c1}) - g(v_{c1})\right] \\ \frac{dv_{c2}}{dt} &= \frac{1}{C_2}\left[\frac{1}{R}(v_{c1} - v_{c2}) + i_L\right] \\ \frac{di_L}{dt} &= \frac{1}{L}[-v_{c2} - R_0 i_L] \end{aligned}\right\} \quad (10)$$

where

$$g(v_{c1}) = av_{c1} + cv_{c1}^3 \quad (11)$$

Fig. 5(a)–(r) shows the bifurcation sequence with respect to R and the chaotic attractors observed experimentally from our experimental setup , including the time waveforms and the spectra of voltages v_{c1}

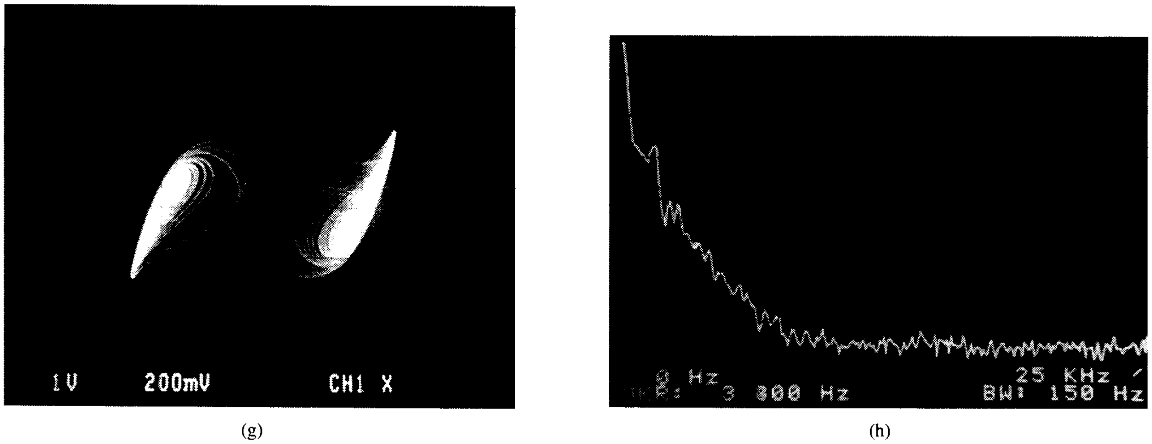


Fig. 6. *Cont.*

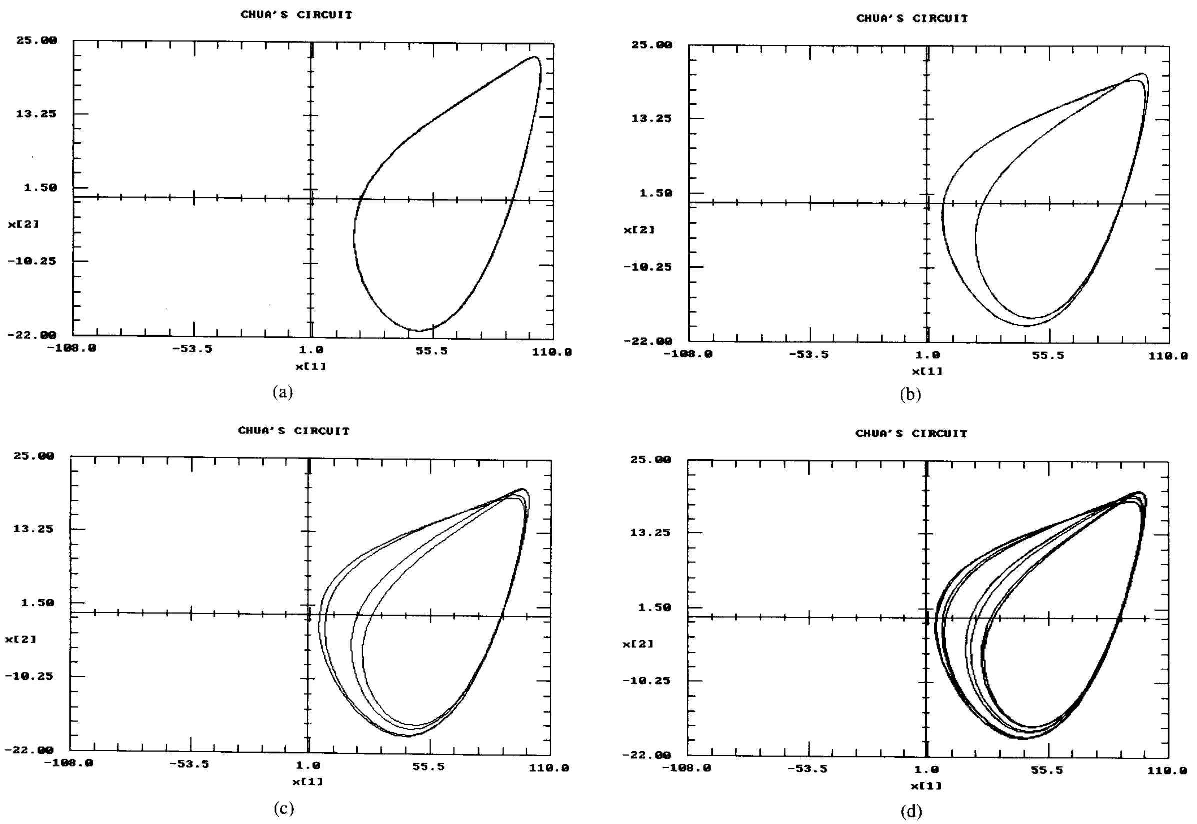
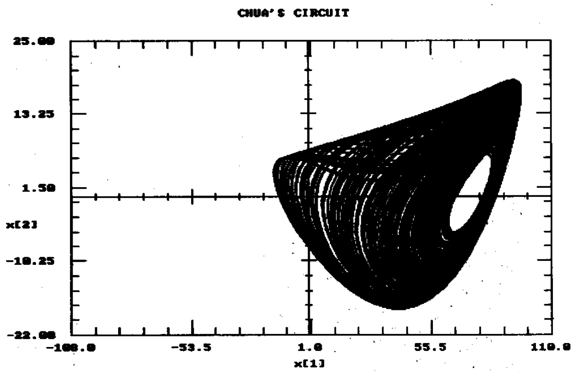


Fig. 7. Simulated bifurcation sequence with respect to the parameter R. Phase portraits in $v_{c1} - v_{c2}$ plane. Horizontal axis v_{c1}. Vertical axis v_{c2}. Parameter values: $C_1 = 7nF, C_2 = 78nF, L = 18.91mH, R_0 = 14.99\Omega$ (the internal resistance of the inductor L), $a = -0.59mS, c = 0.02mS/V^2$. (a) $R = 2200\Omega$, period-1 limit cycle. (b) $R = 2140\Omega$, period-2 limit cycle. (c) $R = 2134\Omega$, period-4 limit cycle. (d) $R = 2131\Omega$, period-8 limit cycle. (e) $R = 2083\Omega$, spiral Chua's attractor. (f) $R = 1964\Omega$, Double-Scroll Chua's attractor.

relative to these phase portraits. Note from these oscilloscope pictures that there is a period-doubling route to chaos similar to that observed from Chua's circuit with a piecewise-linear function.

By adjusting parameters C_1, C_2, and L, a similar bifurcation phenomenon can also be observed, respectively. As an example, we present the bifurcation sequence with respect to capacitor C_2 in Fig. 6(a)–(h). It can be noted from these observations that there is a much wider range of the bifurcation with respect to C_2, e.g., a Double Scroll Chua's attractor can still be observed when C_2 increases up to $600nF$, as shown in Fig. 6(g). In this case the components of high frequencies in Fig. 6(h) are reduced, as expected. This feature will probably be of interest in sound synthesis.



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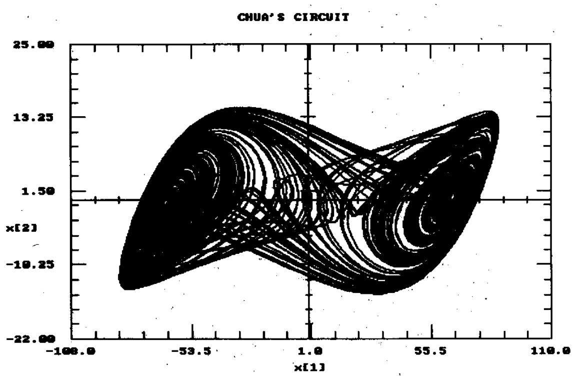

(f)

Fig. 7. *Cont.*

In addition, we do some simulations of this smooth model using the software INSITE. The periodic orbits and chaotic attractors are presented in Fig. 7(a)–(f). Note that the simulations confirm completely our experimental observations.

IV. Concluding Remarks

It is well known that Chua's circuit can exhibit a wide variety of nonlinear behaviors, it has become an attractive paradigm for experimental investigation of chaotic dynamical systems. Though most of the interesting chaotic phenomena can be described by Chua's circuit with a piecewise-linear Chua's diode, some subtle features of the real circuit may be missed by the piecewise-linear approximation. The implementation of a smooth nonlinearity with a cubic polynomial (with even higher order terms) presented in this paper contributes a robust model, with which a more complete experimental model of Chua's circuit can be used for experimental investigations. This model is robust and can be easily integrated in a chip. Furthermore, this method can also be used to design Chua's diodes with almost any *smooth* nonlinearity.

Acknowledgment

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The Use of Parasitic Nonlinear Capacitors in Class E Amplifiers

Michael J. Chudobiak

***Abstract*—The most common class E amplifier configuration uses a single transistor with a shunt capacitor and a series resonant output filter. Until now a linear shunt capacitance has been assumed. However, to achieve operation at 900 MHz and above, it is of interest to rely solely upon the nonlinear parasitic collector-substrate capacitance of the transistor. An analytical theory for operation at 50% duty cycle and nonlinear capacitance is presented in this correspondence, and the effects on the power capability of the amplifier are discussed.**

I. Introduction

Class E tuned power amplifiers have gained widespread acceptance since their introduction [1] due to their simplicity, high efficiency, excellent designability, and relative intolerance to circuit variations [2]. Fig. 1 shows the most common class E configuration.

The transistor acts as a switch, rather than as an amplifier. When the transistor switch is closed, the collector voltage ideally is zero, and a large collector current can exist. When the switch is open, no current flows, but a large collector voltage can exist. Thus, simultaneous nonzero voltage and current is avoided, eliminating transistor power losses in the fully-open and fully-closed states. The capacitor C_1 acts to hold the collector voltage v_c at zero volts during the on-to-off switch transition, to avoid switching losses. The C_F, L_F, jX, R network is designed such that the collector voltage falls back to zero just before the off-to-on transition, again to avoid switching losses. Typical collector or drain voltage and current waveforms for an optimally tuned class E amplifier are shown in Fig. 2.

This circuit has been extensively analyzed [1]–[11], however these analyses have all assumed that the shunt capacitance was constant. As operating frequencies reach 900 MHz and beyond, the shunt capacitance predicted by these analyses may become comparable to the parasitic collector-to-substrate capacitance of the transistor. For

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The author is with Avtech Electrosystems Ltd., Ogdensburg, NY 13669 USA.

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