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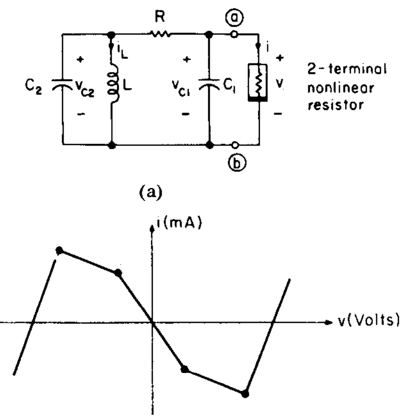


Fig. 1. (a) Chua's circuit. (b) $v-i$ characteristic of nonlinear resistor.

# Periodicity and Chaos in Chua's Circuit

GUO-QUN ZHONG AND F. AYROM

*Abstract* —**This paper reports a period-doubling route to chaos as observed from a laboratory model of the *simplest* possible chaotic autonomous circuit: it is made of two linear capacitors, one linear inductor, one linear resistor, and only one *nonlinear* 2-terminal resistor characterized by a 5-segment piecewise-linear $v-i$ characteristic.**

The circuit shown in Fig. 1(a) with the nonlinear resistor $v-i$ characteristic shown in Fig. 1(b) was recently conceived and proposed by Chua to be the simplest, third-order, autonomous (no ac sources), and reciprocal[1] circuit that could give rise to complicated chaotic dynamics. Several chaotic attractors had since been observed by Matsumoto via *computer simulation* of this circuit over a rather robust parameter range [1]. An experimental confirmation of these attractors has already been reported [2]. Our objective in this paper is to report some bifurcation and chaotic phenomena as *measured* from an actual circuit. Since the nonlinear resistor in Chua's circuit is not available as an "off-the shelf" device, our first task was to design and build such a device. The final circuit shown in Fig. 2(a) uses two operational amplifiers (op amp) and 6 linear resistors (in addition to the standard power supply for the op amp). The $v-i$ characteristics in Figs. 2(b) and (c) are traced from this circuit with two different sets of resistor values (see figure caption). Our experiments reported in this letter are based on measurements obtained by connecting each of these two setups in place of the 2-terminal nonlinear resistor in Fig. 1(a). Note that we have exploited and made full use of the intrinsic *saturation* characteristic of the op amp (normally shunned in standard op amp circuit design) to realize the 5-segment piecewise-linear $v-i$ characteristic in Fig. 1(a); no other nonlinear device is used. From a circuit-theoretic point of view the op amp circuit in Fig. 2(a) should be enclosed by a box with only two terminals ⓐ–ⓑ accessible for external connec-

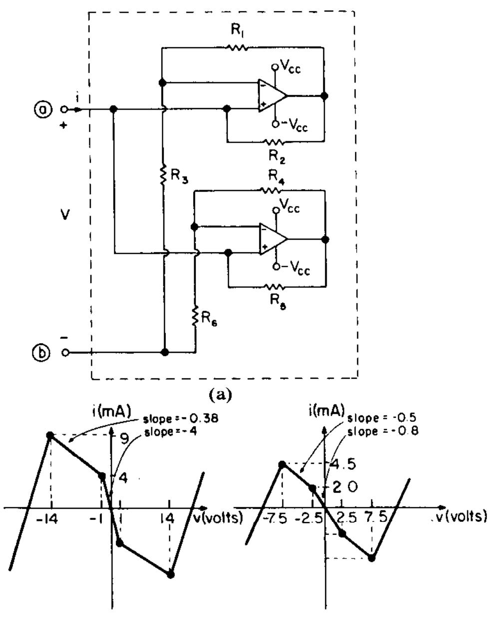


Fig. 2. (a) By adjusting the resistor values, any prescribed 5-segment piecewise-linear $v-i$ characteristic similar to Fig. 1(b) can be realized with this circuit (op amp: National/8035 741LN). (b) $v-i$ characteristic obtained with $V_{cc}=18$ V, $R_1=376\ \Omega$, $R_2=78\ \Omega$, $R_3=5.98$ k$\Omega$, $R_4=312\ \Omega$, $R_5=1.91$ k$\Omega$, and $R_6=52\ \Omega$. (c) $v-i$ characteristic obtained with $V_{cc}=15$ V, $R_1=3.67$ k$\Omega$, $R_2=1.09$ k$\Omega$, $R_3=5.43$ k$\Omega$, $R_4=104\ \Omega$, $R_5=5.36$ k$\Omega$, and $R_6=128\ \Omega$.

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G.-Q. Zhong was a Visiting Scholar at the University of California, Berkeley, CA 94720. He is now with Guangzhou Research Institute of Electronic Technology, Academia Sincia, Guangzhou, People's Republic of China.
F. Ayrom is with the Department of Electrical Engineering and Computer Sciences and the Electronic Science Laboratory, University of California, Berkeley, CA 94720.

[1]From a circuit theory viewpoint, a *reciprocal* circuit is one made of only *two*-terminal elements such as resistors, inductors, capacitors, batteries, diodes, etc. Circuits containing transistors are, therefore, nonreciprocal and are generally considered to be more complicated.

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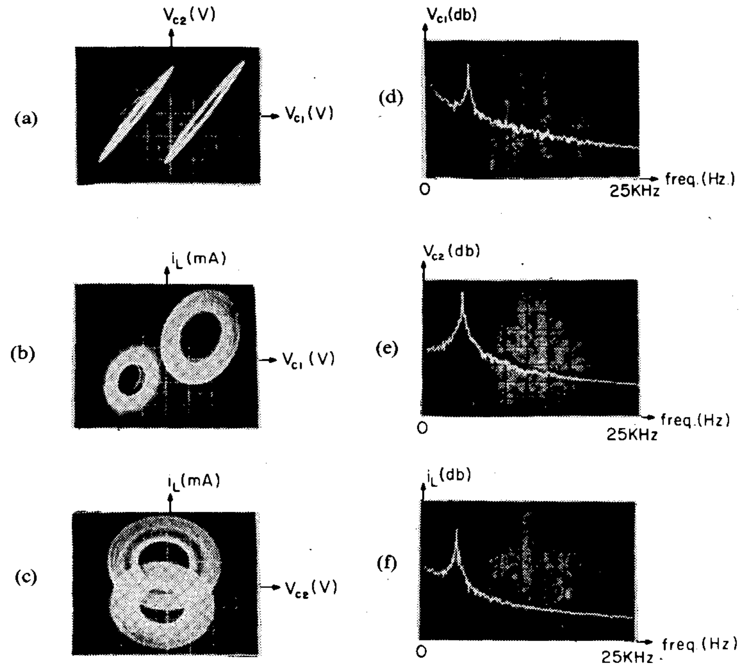


Fig. 3. Chaotic attractors measured from Chua's circuit using the nonlinear device of Fig. 2(b) and with $R = 1.03$ kΩ, $C_1 = 0.005$ ωF, $C_2 = 0.1$ μF, and $L = 7.6$ mH. (a) Chaotic attractor in the $V_{C1} - V_{C2}$ plane: scale: $V_{C1} = 4$V/div, $V_{C2} = 2$ V/div. (b) Chaotic attractor in the $V_{C1} - i_L$ plane: scale: $i_L = 6$ ma/div, $V_{C1} = 4$ V/div. (c) Chaotic attractor in the $V_{C2} - i_L$ plane: scale: $i_L = 6$ ma/div, $V_{C2} = 2$ V/div. (d) Spectrum of $V_{C1}(t)$. (e) Spectrum of $V_{C2}(t)$. (f) Spectrum of $i_L(t)$.

tion, and is therefore to be classified as a 2-terminal device.

Compared to the other *autonomous* chaotic circuits reported in the literature [3], [4], Chua's circuit is the simplest possible in the sense that chaos can not occur in an autonomous circuit (modeled by nonlinear state equations), with fewer than 3 energy storage elements (capacitors and inductors) and that at least one nonlinear element is needed even for oscillation to be possible.

We have built the circuit in Fig. 1(a) and observed a great variety of bifurcation and chaotic phenomena from different combinations of circuit parameters ($R$, $L$, $C_1$, and $C_2$) as well as different choices of the nonlinear $v-i$ characteristics (breakpoints coordinates and slopes). In this paper we summarize the phenomena observed from two set-ups corresponding to the two $v-i$ characteristics shown in Fig. 2(b) and (c), respectively.

Connecting terminals ⓐ–ⓑ of the circuit in Fig. 2(a) in place of the nonlinear resistor in Fig. 1(a), various Lissajous figures and spectrums are measured and shown in Figs. 3 and 4, respectively. The circuit parameters and scales used in these oscilloscope tracings are given in the figure captions.

Fig. 3(a)–(c) gives 2 perspectives of the same chaotic attractor. It consists of two rings joined at the upper and the lower edge by a thin sheet of ribbon made of trajectories. A computer simulation of this circuit (not shown) gives rise to a virtually identical attractor and reveals that each trajectory winds around each ring in a counterclockwise direction and grow in size until it hits some boundary set near the outer edge of the ring, whereupon it exits *rapidly* along a thin ribbon and lands near the center of the other ring, thereby repeating the above phenomenon. The exit points and times are observed to be random, thereby accounting for the "double-ring" limiting set $\mathscr{S}$. Since trajectories originating from nearby points outside of $\mathscr{S}$ are quickly attracted to $\mathscr{S}$, it is natural to call $\mathscr{S}$ a *chaotic* attractor. The spectrums corresponding to the voltage waveform $V_{C1}(t)$ across capacitor $C_1$, the voltage waveform $V_{C2}(t)$ across capacitor $C_2$ and the current waveform $i_1(t)$ through the inductor $L$ as shown in Fig. 3(d), (e), and (f), respectively, are seen to resemble a broad-spectrum noise.

Fig. 4(a)–(e) displays the period-doubling route to chaos in Chua's circuit using the nonlinearity of Fig. 2(c). The single-loop limit cycle in Fig. 4(a) represents a well-defined periodic waveform spawned by a Hopf bifurcation process when a pair of complex-conjugate eigenvalues associated with an equilibrium point of this circuit crosses the imaginary axis and enters the right-half plane. A slight decrease in the value of $R$ leads to the sequence of Lissajous figures shown in Fig. 4(b)–(e). Fig. 4(a)–(d) shows three successive period-doublings giving rise to a 2, 4, and 8-loop limit cycles as we continue to decrease $R$ in small amounts. A further small decrease in $R$ leads to the chaotic attractor in Fig. 4(e). The structure between the two rings in this attractor is clearer than that shown in Fig. 3(a)–(c). It appears also to be quite a bit more complicated than is apparent from the earlier thin ribbon paths. Since the range of $R$ between Fig. 4(a) and 4(e) is very narrow (1.7–1.5 Ω), as is typical of the convergence property of the period-doubling bifurcation parameter, we were unable to observe a periodic waveforms with order higher than 8. Our results, however, suggest strongly that the chaotic attractor in Fig. 4(e) is spawned by a period-doubling mechanism.

Finally, we remark that one of the reviewers pointed out that in [5] a second-order circuit has been presented which exhibits chaos. However, the nonlinearity in this system is described by a binary hysteresis (which is a multivalued dynamic nonlinearity) and *can not* be described by a second order state equation $\dot{x} = f(x)$ with a *continuos* $f(\cdot)$. In other words Chua's circuit is indeed the simplest reciprocal autonomous circuit capable of exhibiting chaotic behavior.

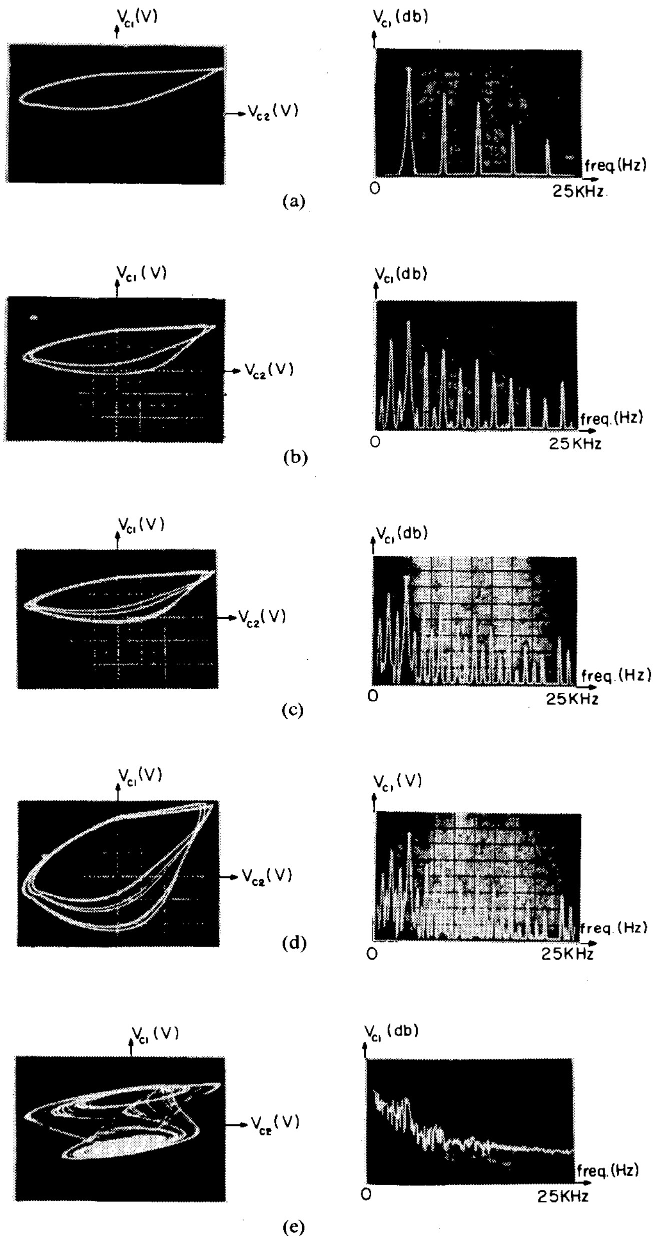


Fig. 4. Waveforms (left side) and spectrums (right side) showing the period doubling route to chaos in Chua's circuit using the nonlinear device of Fig. 2(c) and with $C_1 = 0.005\ \mu$F, $C_2 = 0.05\ \mu$F and $L = 7.2$ mH. (a) Single-loop cycle. Scale: $V_{C1} = 5$ V/div, $V_{C2} = 0.4$ V/div. (b) Double-loop cycle. Scale: $V_{C1} = 5$ V/dib, $V_{C2} = 0.4$ V/div. (c) 4-loop cycle. Scale: $V_{C1} = 5$ V/div, $V_{C2} = 0.4$ V/div. (d) 8-loop cycle. Scale: $V_{C1} = 2$ V/div, $V_{C2} = 0.4$ V/div. (e) Chaotic attractor in the $V_{C2}-V_{C1}$. Scale: $V_{C1} = 5$ V/div, $V_{C2} = 0.4$ V/div.

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# A Note on Global Implicit Function Theorems

SHIGEO ICHIRAKU

***Abstract*—Applying the theory of covering maps, we will give an extension of global implicit function theorems proved by Sandberg [1]. Although he mentioned the theory of covering maps, he did not appeal to it. In fact,**

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The author is with the Department of Mathematics, Yokohama City University, Yokohama, Japan 236.

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