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IMPULSIVE SYNCHRONIZATION OF CHAOTIC LUR'E SYSTEMS BY MEASUREMENT FEEDBACK

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In this paper we consider impulsive control of master-slave synchronization schemes that consist of identical Lur'e systems. Impulsive control laws are investigated which make use of linear or nonlinear dynamic measurement feedback. A sufficient condition for global asymptotic stability is presented which is characterized by a set of matrix inequalities. Synchronization is proven for the error between the output signals. The method is illustrated on Chua's circuit and a hyperchaotic system with coupled Chua's circuits.

1. Introduction

Recently, methods for synchronization of nonlinear systems have been proposed which make use of impulsive control laws [Yang & Chua, 1997a, 1997b; Yang *et al.*, 1997; Stojanovski *et al.*, 1996, 1997]. In this way the error system of the synchronization scheme is stabilized using small control impulses. These methods are offering a direct method for modulating digital information onto a chaotic carrier signal for spread spectrum applications [Wu & Chua, 1997] and has been applied to chaotic digital code-division multiple access (CDMA) systems in [Yang & Chua, 1997c]. The method discussed in [Yang & Chua, 1997a, 1997b; Yang *et al.*, 1997] is based on a theory of impulsive differential equations described in [Lakshmikantham *et al.*, 1989]. At discrete time instants, jumps in the system's state are caused by a control input. Global asymptotic stability of the error system is proven by means of a Lyapunov function and is characterized by a set of conditions related to the time instants, the time intervals in between and a coupling condition between these.

However, so far the method has been applied ad hoc to the special cases of Chua's circuit [Yang & Chua, 1997a, 1997b] and the Lorenz system [Yang *et al.*, 1997]. Moreover full state information has been assumed, which means that knowledge of the full state vector of the system is needed in order to synchronize the systems by impulses. The aim of this paper is to present a general design procedure for master-slave synchronization schemes

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which consist of identical Lur'e systems [Khalil, 1992; Suykens *et al.*, 1996; Vidyasagar, 1993]. Examples of chaotic and hyperchaotic Lur'e systems are Chua's circuit [Chua *et al.*, 1986; Chua, 1994; Madan, 1993], generalized Chua's circuits that exhibit n-scroll attractors [Suykens *et al.*, 1997b] and arrays which consist of such chaotic cells [Kapitaniak & Chua, 1994; Suykens & Chua, 1997]. In practice the full state vector is often not available, not measurable or too expensive to measure. Therefore we investigate the case of measurement feedback for which we derive a sufficient condition of global asymptotic stability of the error system. This error is defined between the outputs (instead of states) of the master and slave system. For the sake of generality we study a nonlinear dynamic output feedback law where the state equation of the controller takes the form of a Lur'e system. This impulsive control law includes static output feedback and linear dynamic output feedback as special cases. The latter method has been discussed in [Suykens *et al.*, 1997a] for a continuous control law. The conditions for synchronization have been expressed as a matrix inequalities [Boyd *et al.*, 1994], which also occur in the context of nonlinear H_∞ synchronization methods for secure communications applications [Suykens *et al.*, 1997c]. The design of the controller is done then by solving a nonlinear optimization problem which involves the matrix inequality. A similar approach is followed in this paper for the impulsive control case.

We illustrate the method on Chua's circuit and a hyperchaotic system with coupled Chua's circuits that exhibits the double–double scroll attractor [Kapitaniak & Chua, 1994]. Two identical Chua's circuits are impulsively synchronized by a linear dynamic output controller of first order where one state variable is measured on the circuits and one single control input is taken. While synchronization is theoretically proven for the difference between the measured state variables, the complete state vectors are synchronizing as well according to the simulation results. In another example it is shown how two double–double scroll attractors can be synchronized by measuring only one state variable and taking one single control input. In this case however, the synchronization is occurring for the output but not for the full state vector. Synchronization for the full state vector is obtained by a linear dynamic output feedback controller with two outputs and two control inputs (one for each of the two cells).

This paper is organized as follows. In Sec. 2 we present the master–slave synchronization scheme with impulsive control. In Sec. 3 the matrix inequalities are derived and controller design is discussed. In Sec. 4 examples are given.

2. Synchronization Scheme

We consider the following master–slave synchronization scheme

$$
\begin{aligned}
&\mathcal{M}: \begin{cases} \dot{x} = Ax + B\sigma(Cx) \\ p = Lx \end{cases} \\
&\mathcal{S}: \begin{cases} \dot{z} = Az + B\sigma(Cz), & t \neq \tau_i \\ q = Lz \end{cases} \\
&\mathcal{C}: \begin{cases} \dot{\xi} = E\xi + F(p-q) + W_F\sigma(V_{F_1}\xi + V_{F_2}(p-q)), & t \neq \tau_i \\ \Delta z = D_1 u, & t = \tau_i \\ \Delta\xi = D_2 v, & t = \tau_i \\ u = G_1\xi + H_1(p-q) \\ v = G_2\xi + H_2(p-q) \end{cases}
\end{aligned}
\tag{1}
$$

which consists of master system $\mathcal{M}$, slave system $\mathcal{S}$ and controller $\mathcal{C}$. $\mathcal{M}$ and $\mathcal{C}$ are identical Lur'e system with state vectors x, $z \in \mathbb{R}^n$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_h}$, $C \in \mathbb{R}^{n_h\times n}$. A Lur'e system is a linear dynamical system, feedback interconnected to a static nonlinearity $\sigma(\cdot)$ that satisfies a sector condition [Khalil, 1992; Vidyasagar, 1993] (here it has been represented as a recurrent neural network with one hidden layer, activation function $\sigma(\cdot)$ and n_h hidden units [Suykens *et al.*, 1996]). We assume that $\sigma(\cdot) : \mathbb{R}^{n_h} \mapsto \mathbb{R}^{n_h}$ is a diagonal nonlinearity with $\sigma_i(\cdot)$ belonging to sector $[0, k]$ for $i = 1, \ldots, n_h$. The output (or measurement) vectors of $\mathcal{M}$ and $\mathcal{S}$ are $p, q \in \mathbb{R}^l$ with $l \leq n$ and $L \in \mathbb{R}^{l \times n}$.

For the impulsive control law $\mathcal{C}$, a set of discrete time instants τ_i is considered where $0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots$ with $\tau_i \to \infty$ as $i \to \infty$ [Lakshmikantham *et al.*, 1989; Yang & Chua, 1997a, 1997b; Yang *et al.*, 1997]. For the sake of generality, a nonlinear dynamic output feedback controller of Lur'e form is taken here for the state equation with state vector $\xi \in \mathbb{R}^{n_\xi}$. At the time instants τ_i, jumps in the state variables z and ξ are imposed

$$\begin{aligned} \Delta z|_{t=\tau_i} &= z(\tau_i^+) - z(\tau_i^-) \\ \Delta \xi|_{t=\tau_i} &= \xi(\tau_i^+) - \xi(\tau_i^-) \,. \end{aligned} \tag{2}$$

By means of the matrices D_1 and D_2, the state equations on which the impulsive controls $u \in \mathbb{R}^{m_z}$ and $v \in \mathbb{R}^{m_\xi}$ are applied, are decided upon. The output difference $p - q$ is taken as input of the controller $\mathcal{C}$. The matrices of the controller are of dimension $E \in \mathbb{R}^{n_\xi \times n_\xi}$, $F \in \mathbb{R}^{n_\xi \times l}$, $W_F \in \mathbb{R}^{n_\xi \times n_{h_\xi}}$, $V_{F_1} \in \mathbb{R}^{n_{h_\xi} \times n_\xi}$, $V_{F_2} \in \mathbb{R}^{n_{h_\xi} \times l}$, $D_1 \in \mathbb{R}^{n_z \times m_z}$, $D_2 \in \mathbb{R}^{n_\xi \times m_\xi}$, $G_1 \in \mathbb{R}^{m_z \times n_\xi}$, $G_2 \in \mathbb{R}^{m_\xi \times n_\xi}$, $H_1 \in \mathbb{R}^{m_z \times l}$, $H_2 \in \mathbb{R}^{m_\xi \times l}$ where n_{h_ξ} is the number of hidden units in the Lur'e system of $\mathcal{C}$. Note that the control law also includes the cases of static output feedback ($G_1 = 0$, $G_2 = 0$) and linear dynamic output feedback ($W_F = 0$, $V_{F_1} = 0$, $V_{F_2} = 0$).

Given the synchronization scheme (1), the synchronization error is defined as $e = x - z$ for the state vectors and $e_L = p - q = Le$ for the outputs. The first case yields the error system

$$\mathcal{E}_1 : \begin{cases} \dot{e} &= Ae + B\eta(Ce;\, z), & t \neq \tau_i \\ \dot{\xi} &= E\xi + F(p-q) + W_F\sigma(V_{F_1}\xi + V_{F_2}(p-q)), & t \neq \tau_i \\ \Delta e &= -D_1 u, & t = \tau_i \\ \Delta \xi &= D_2 v, & t = \tau_i \\ u &= G_1\xi + H_1(p-q) \\ v &= G_2\xi + H_2(p-q) \end{cases} \tag{3}$$

where $\eta(Ce;\, z) = \sigma(Ce + Cz) - \sigma(Cz)$ and $\Delta e = \Delta x - \Delta z$ with $\Delta x = 0$ for the master system. The error system for e_L becomes

$$\mathcal{E}_2 : \begin{cases} \dot{e}_L &= LAe + LB\eta(Ce;\, z), & t \neq \tau_i \\ \dot{\xi} &= E\xi + F(p-q) + W_F\sigma(V_{F_1}\xi + V_{F_2}(p-q)), & t \neq \tau_i \\ \Delta e_L &= -LD_1 u, & t = \tau_i \\ \Delta \xi &= D_2 v, & t = \tau_i \\ u &= G_1\xi + H_1(p-q) \\ v &= G_2\xi + H_2(p-q) \end{cases} \tag{4}$$

with $\Delta e_L = -L\Delta z$. This scheme will be studied in the sequel.

3. Stability, Matrix Inequalities and Controller Design

In order to derive a sufficient condition for global asymptotic stability of the error system $\mathcal{E}_2$, we take the Lyapunov function

$$V(e_L, \xi) = \zeta^T P \zeta = [e_L^T \quad \xi^T] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} e_L \\ \xi \end{bmatrix}, \quad P = P^T > 0 \,. \tag{5}$$

According to Lakshmikantham *et al.* [1989], Yang and Chua [1997a, 1997b] and Yang *et al.* [1997] it is sufficient then to prove that

$$\begin{cases} \dot{V} \le \alpha V, & \alpha > 0, \quad t \ne \tau_i & \text{(6a)} \\ V(\zeta + \Delta\zeta) < \beta V, & \beta > 0, \quad t = \tau_i & \text{(6b)} \\ \|\zeta + \Delta\zeta\|_2 < \|\zeta\|_2, & \quad t = \tau_i & \text{(6c)} \\ \alpha(\tau_{i+1} - \tau_i) + \log\beta < 0\,. & & \text{(6d)} \end{cases}$$

From Eq. (6d) we find that $\beta < 1$ should be satisfied. We will express the conditions (6a)–(6c) now as matrix inequalities. In the derivation we exploit the inequalities

$$\begin{cases} \eta(Ce)^T \Lambda[\eta(Ce) - Ce] \le 0, & \forall e \in \mathbb{R}^n \\ \sigma(\varphi)^T \Gamma[\sigma(\varphi) - V_{F_1}\xi - V_{F_2}Le] \le 0, & \forall e \in \mathbb{R}^n,\ \xi \in \mathbb{R}^{n_\xi}\,. \end{cases} \tag{7}$$

These are related to the sector conditions on the nonlinearities $\eta(\cdot)$ and $\sigma(\cdot)$, which are assumed to belong to sector [0, 1]. Λ and Γ are diagonal matrices with positive diagonal elements and $\varphi = V_{F_1}\xi + V_{F_2}Le$. By employing (7) in an application of the S-procedure [Boyd *et al.*, 1994] a matrix inequality is obtained by writing

$$\dot{V} - \alpha V - 2\eta(Ce)^T \Lambda[\eta(Ce) - Ce] - 2\sigma(\varphi)^T \Gamma[\sigma(\varphi) - V_{F_1}\xi - V_{F_2}Le] \le 0 \tag{8}$$

as a quadratic form $w^T Z w \le 0$ in $w = [e;\ \xi;\ \eta;\ \sigma]$. Imposing this quadratic form to be negative semidefinite for all w, one obtains

$$Z = Z^T = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ \cdot & Z_{22} & Z_{23} & Z_{24} \\ \cdot & \cdot & Z_{33} & 0 \\ \cdot & \cdot & \cdot & Z_{44} \end{bmatrix} \le 0 \tag{9}$$

with

$$\begin{aligned} Z_{11} &= A^T P_{11} L + L^T P_{11} A + L^T P_{12} F L & Z_{12} &= A^T P_{12} + L^T P_{12} E + L^T F^T P_{22} - \alpha L^T P_{12} \\ &\quad + L^T F^T P_{21} L - \alpha L^T P_{11} L & Z_{13} &= L^T P_{11} L B + C^T \Lambda \\ Z_{22} &= E^T P_{22} + P_{22} E - \alpha P_{22} & Z_{14} &= L^T P_{12} W_F + L^T V_{F_2}^T \Gamma \\ Z_{33} &= -2\Lambda & Z_{23} &= P_{21} L B \\ Z_{44} &= -2\Gamma & Z_{24} &= P_{22} W_F + V_{F_1}^T \Gamma\,. \end{aligned}$$

In order to express the other conditions (6b) and (6c) as matrix inequalities, we write

$$\zeta + \Delta\zeta = \begin{bmatrix} e_L \\ \xi \end{bmatrix} + \begin{bmatrix} \Delta e_L \\ \Delta\xi \end{bmatrix} = M \begin{bmatrix} e_L \\ \xi \end{bmatrix} \tag{10}$$

with

$$M = \begin{bmatrix} I - LD_1H_1 & -LD_1G_1 \\ D_2H_2 & I + D_2G_2 \end{bmatrix}.$$

This yields the matrix inequality

$$M^T P M < \beta P \tag{11}$$

for (6b) and

$$M^T M < I \tag{12}$$

for (6c). The latter matrix inequality is the underlying reason why we derived synchronization criteria for the error system $\mathcal{E}_2$ instead of $\mathcal{E}_1$, because it turns out that for $\mathcal{E}_1$ the condition (6c) leads to infeasibility.

The controller design is based then on the matrix inequalities (9), (11) and (12) by solving the feasibility problem

$$\begin{aligned} &\text{Find} \quad \theta_c,\ Q,\ \Lambda,\ \Gamma,\ \alpha,\ \beta \\ &\text{such that} \quad \begin{cases} Z \le 0 \\ M^T P M < \beta P \\ M^T M < I \\ \alpha(\tau_{i+1} - \tau_i) + \log\beta < 0 \end{cases} \end{aligned} \tag{13}$$

with $P = Q^T Q$ and the controller parameter vector θ_c containing the elements of the matrices E, F, W_F, V_{F_1}, V_{F_2}, G_1, H_1, G_2, H_2. This problem has to be solved for given matrices A, B, C, L, D_1, D_2 and a fixed choice of the time interval $\tau_{i+1} - \tau_i$.

4. Examples

In this section we illustrate the method on Chua's circuits and coupled Chua's circuits that exhibit the double–double scroll attractor.

4.1. *Chua's circuit*

We consider master–slave synchronization of two identical Chua's circuits by means of impulsive control. We take the following representation of Chua's circuit for the master system $\mathcal{M}$:

$$\begin{cases} \dot{x}_1 = a[x_2 - h(x_1)] \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -bx_2 \end{cases} \tag{14}$$

with nonlinear characteristic

$$h(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)(|x_1 + c| - |x_1 - c|) \tag{15}$$

and parameters $a = 9$, $b = 14.286$, $m_0 = -1/7$, $m_1 = 2/7$ in order to obtain the double scroll

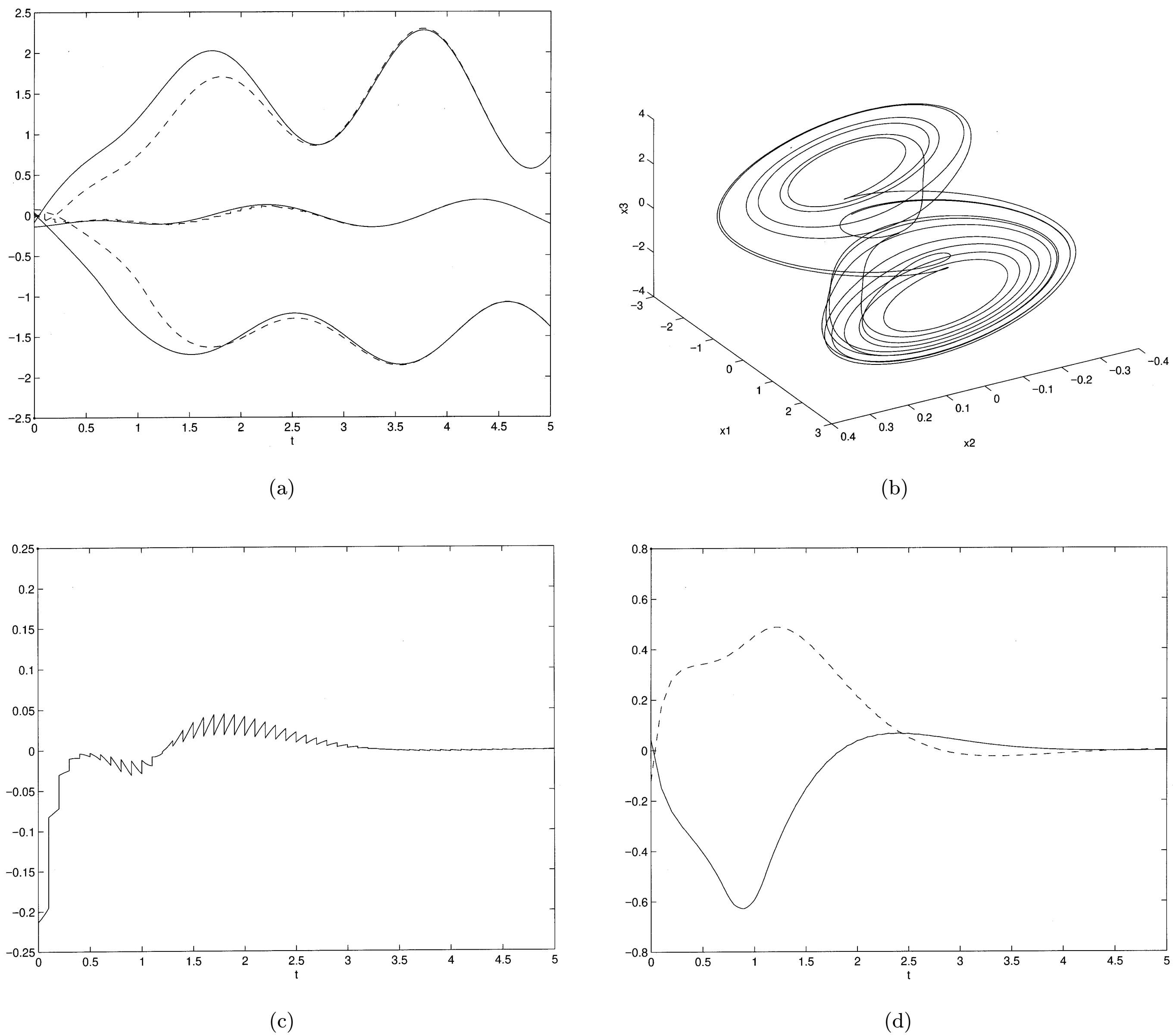
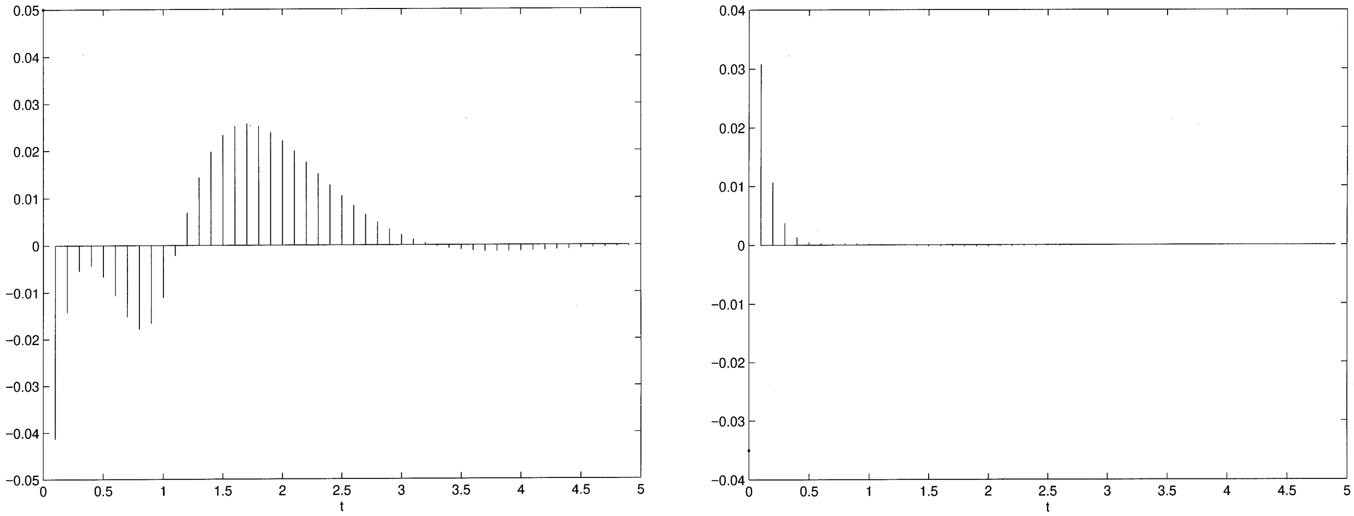
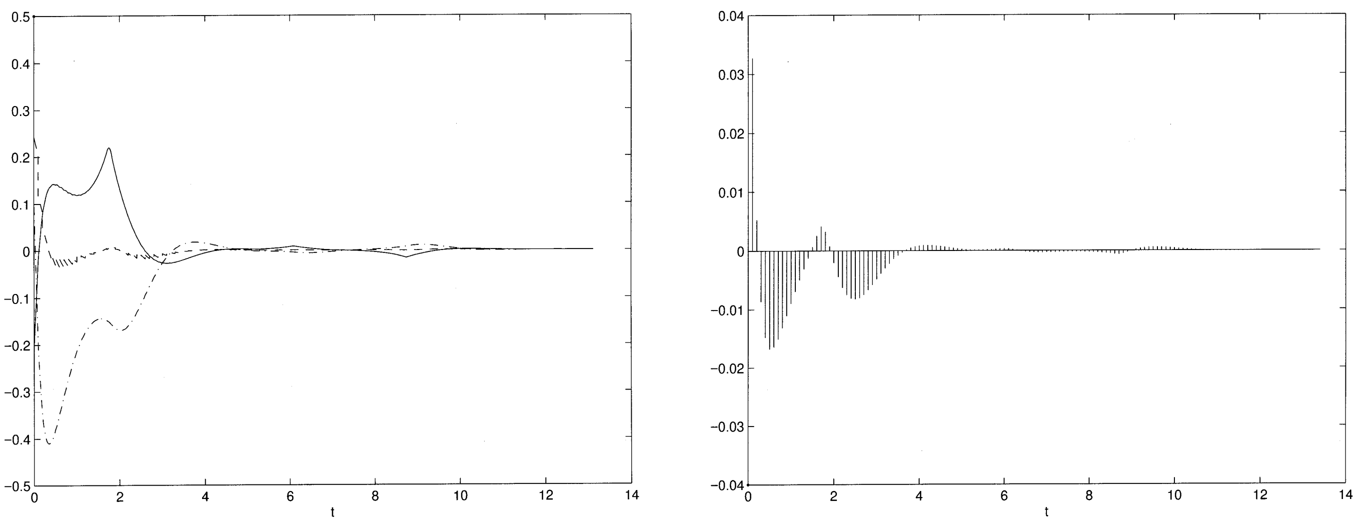


Fig. 1. Synchronization of two Chua's circuits by impulsive linear dynamic output feedback with one output and one control input: (a) $x(t)$ (solid line), $z(t)$ (dashed line); (b) three-dimensional view on the double scroll attractor generated at the master system; (c) output synchronization error $e_L(t) = e_2(t) = x_2 - z_2$; (d) $e_1(t)$ (solid line), $e_3(t)$ (dashed line); (e) impulsive control $D_1 u(t)$ applied to the slave system; (f) impulsive control $D_2 v(t)$ applied to the controller.



(e) (f)

Fig. 1. (*Continued*)



(a) (b)

Fig. 2. Synchronization of two Chua's circuits by impulsive nonlinear dynamic output feedback with one output and one control input: (a) $e_L(t) = e_2(t)$ (dashed line), $e_1(t)$ (solid line), $e_3(t)$ (dash-dotted line); (b) impulsive control $D_1 u(t)$.

attractor [Chua *et al.*, 1986; Chua, 1994; Madan, 1993]. The nonlinearity $\phi(x_1) = (1/2)(|x_1 + c| - |x_1 - c|)$ (linear characteristic with saturation) belongs to sector $[0, 1]$. A Lur'e representation $\dot{x} = Ax + B\phi(Cx)$ of Chua's circuit is given then by

$$A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]. \tag{16}$$

Defining the outputs $p = x_2$, $q = z_2$ (hence $L = [0 \quad 1 \quad 0]$, $l = 1$) and impulsive control with $D_1 = \text{diag}\{0, 1, 0\}$, $D_2 = I$, $n_\xi = 1$, $m_\xi = 1$, $W_{EF} = 0$, $V_{F_1} = 0$, $V_{F_2} = 0$ (linear dynamic output feedback controller) the optimization problem (13) has been solved. Sequential quadratic programming [Fletcher, 1987] by means of the function constr of Matlab has been applied. The first constraint in (13) was used as objective function $\lambda_{\max}(Z)$ (where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue of a symmetric matrix) while the remaining three constraints have been imposed as hard constraints. The following starting points have been chosen for the optimization: θ_c random according to a Gaussian distribution with zero mean and standard deviation 0.1; $Q = I$; $\Lambda = I$; $\alpha = 10$; $\beta = 0.1$. The time interval $\tau_{i+1} - \tau_i$ was chosen fixed and equal to 0.1. Instead of β, the parameter $1/[1+\exp(-\beta)]$ (which belongs to $(0, 1)$) was taken as unknown of the optimization problem. A feasible point which satisfies the constraints and brings the objective function close to zero is shown in Fig. 1. While synchronization is only proven for $x_2 - z_2$ by the Lyapunov function (5), the synchronization error $x - z$ for the complete state vector is also tending to zero. Small control impulses have to be applied to the slave system and to the linear dynamic output feedback controller. Simulations have been done on a SUN Ultra 2 workstation with a Runge–Kutta integration rule (*ode23* in Matlab) with tolerance $1.0e - 10$.

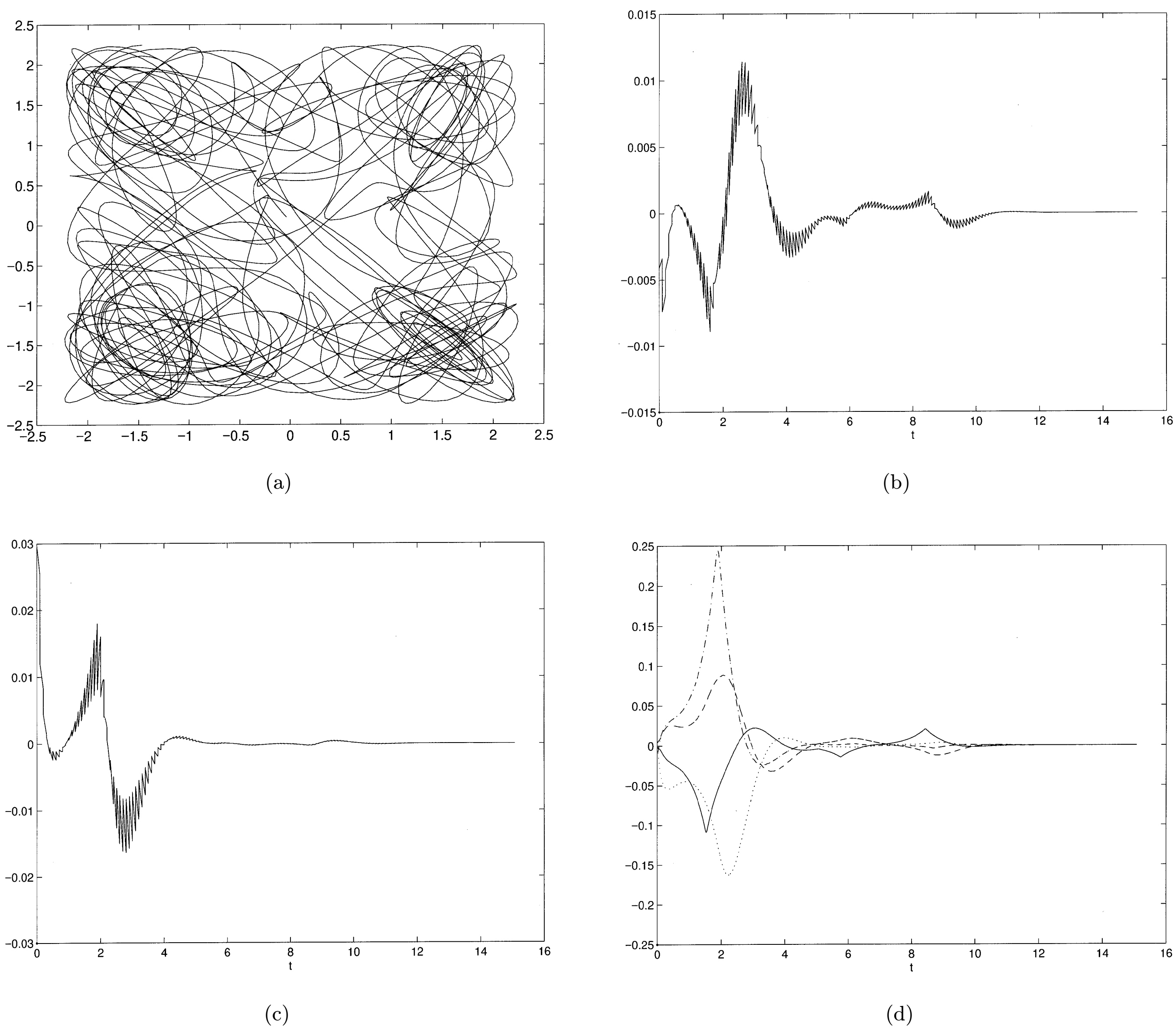
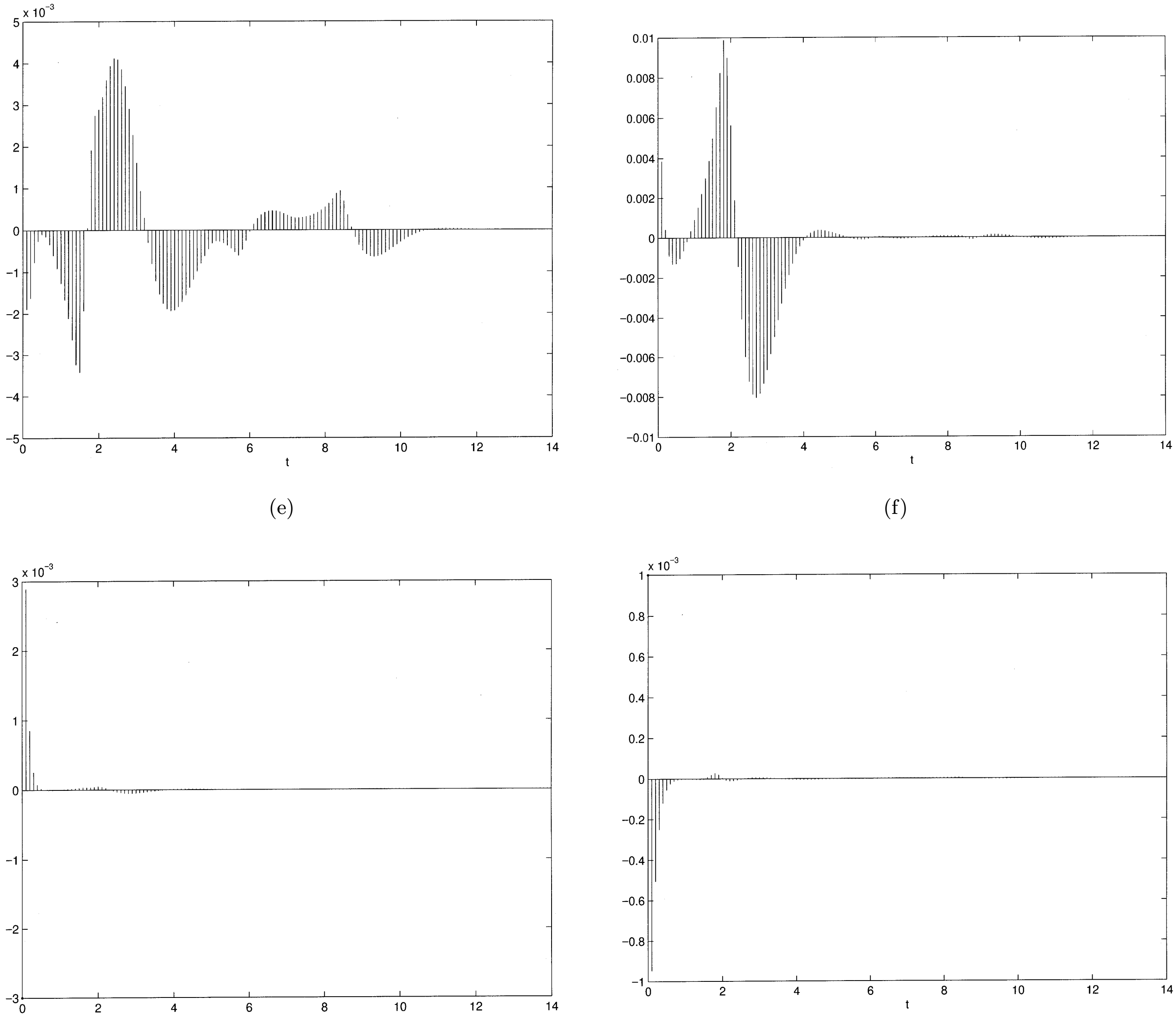


Fig. 3. Synchronization of two hyperchaotic systems (coupled Chua's circuits) by impulsive linear dynamic output feedback with two outputs and two control inputs: (a) double–double scroll attractor according to Kapitaniak & Chua, shown is (x_1, x_4) for this hyperchaotic system with six state variables; (b) output synchronization error $e_2(t) = x_2 - z_2$; (c) output synchronization error $e_5(t) = x_5 - z_5$; (d) $e_1(t)$ (solid line), $e_3(t)$ (dashed line), $e_4(t)$ (dash-dotted line), $e_6(t)$ (dotted line); (e) impulsive control $u_2(t)$ applied to the slave system; (f) impulsive control $u_5(t)$ applied to the slave system; (g) impulsive control $v_1(t)$ applied to the controller; (h) impulsive control $v_2(t)$ applied to the controller.



(e) (f) (g) (h)

Fig. 3. (*Continued*)

A nonlinear dynamic output feedback law (1) has been studied for the same outputs and control input with $D_1 = \text{diag}\{0, 1, 0\}$, $D_2 = I$, $n_\xi = 3$, $m_\xi = 3$, $n_{h_\xi} = 1$ (controller with the same dimensions for A, B, C as Chua's circuit but different values for the matrices). The same parameters were taken as before for the initialization of sequential quadratic programming, together with $\Gamma = I$. Simulation results are shown on Fig. 2. As for the linear feedback case the synchronization error between the full state vectors is tending to zero, while this is theoretically proven only for the measured variables.

4.2. *Coupled Chua's circuits*

We consider the following master system which consists of two unidirectionally coupled Chua circuits [Kapitaniak & Chua, 1994]

$$\begin{cases} \dot{x}_1 = a[x_2 - h(x_1)] \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -bx_2 \\ \dot{x}_4 = a[x_5 - h(x_4)] + K(x_4 - x_1) \\ \dot{x}_5 = x_4 - x_5 + x_6 \\ \dot{x}_6 = -bx_5 \end{cases} \tag{17}$$

with $h(x_i) = m_1 x_i + (1/2)(m_0 - m_1)(|x_i + c| - |x_i - c|)$ $(i = 1, 4)$. For $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.286$, $c = 1$, $K = 0.01$ the system exhibits hyperchaotic behavior with a double–double scroll attractor. The system can be represented in Lur'e form with $n = 6$, $n_h = 2$ and

$$A = \left[\begin{array}{ccc|ccc} -am_1 & a & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -a\,m_1 & a & 0 \\ 0 & -K & 0 & 1 & -1+K & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} -a(m_0 - m_1) & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & -a\,(m_0 - m_1) \\ 0 & 0 \\ 0 & 0 \end{array}\right],$$

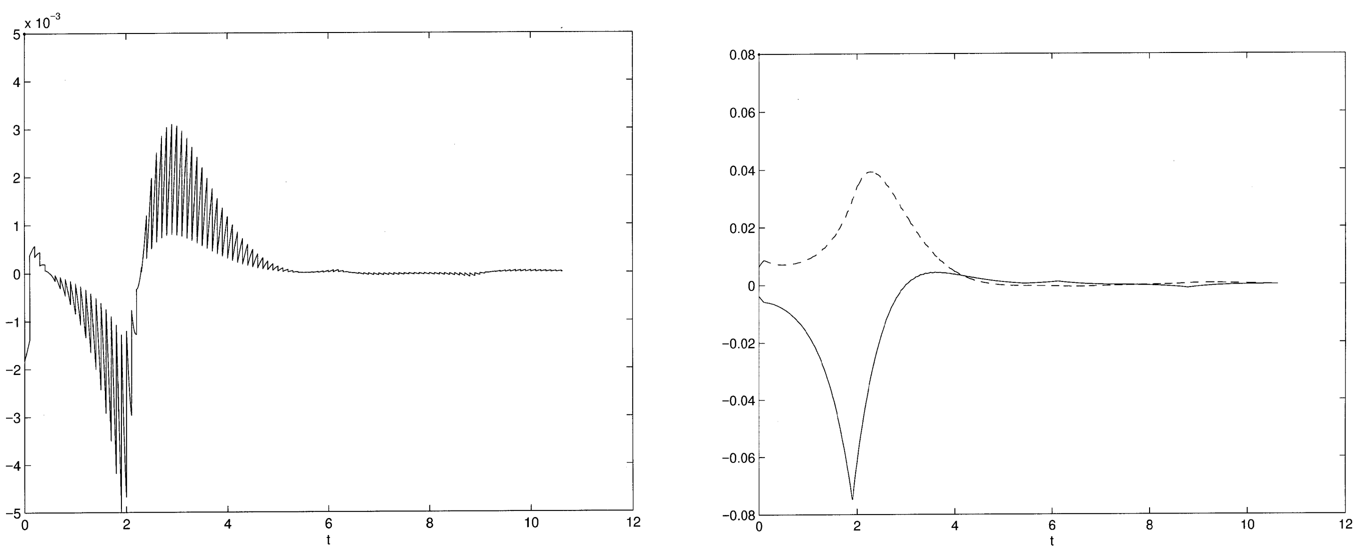
$$C = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]. \tag{18}$$

We investigate the case of linear dynamic output feedback. First we define two outputs for the system ($p = [x_2;\, x_5]$, $q = [z_2;\, z_5]$) and two controls inputs ($D_1 = \mathrm{diag}\{0,\, 1,\, 0,\, 0,\, 1,\, 0\}$). Furthermore we choose $n_\xi = 2$, $m_\xi = 2$, $W_{EF} = 0$, $V_{F_1} = 0$, $V_{F_2} = 0$. The initialization is done in the same way as in the previous examples. Simulation results show (Fig. 3) that the synchronization is obtained for the full state vector of the hyperchaotic system, while it is theoretically only shown for the outputs.

Next we study the case of linear dynamic output feedback with one output ($p = x_2$, $q = z_2$) and one control input ($D_1 = \mathrm{diag}\{0,\, 1,\, 0,\, 0,\, 0,\, 0\}$). The other parameters and initialization were chosen in the same way as in the previous case. Simulation results are shown in Fig. 4. The synchronization error is tending to zero for the difference between the measurements, but not for the full state vector.

5. Conclusions

We discussed a systematic procedure for designing impulsive control laws in order to synchronize Lur'e systems. Examples of chaotic Lur'e systems are (generalized) Chua's circuits and arrays that contain such chaotic cells. The method makes use of measurement feedback instead of full state feedback. For the sake of generality, nonlinear dynamic output feedback controllers have been investigated which include the special cases of static



(a) (b)

Fig. 4. Synchronization of the two hyperchaotic systems of Fig. 3 but with one output and one control input: (a) output synchronization error $e_L(t) = e_2(t) = x_2 - z_2$; (b) $e_1(t)$ (solid line), $e_3(t)$ (dashed line) (first cell); (c) $e_4(t)$ (solid line), $e_5(t)$ (dashed line), $e_6(t)$ (dotted line) (second cell); (d) impulsive control $D_1 u(t)$ applied to the slave system; (f) impulsive control $v_1(t)$ applied to the controller; (g) impulsive control $v_2(t)$ applied to the controller.

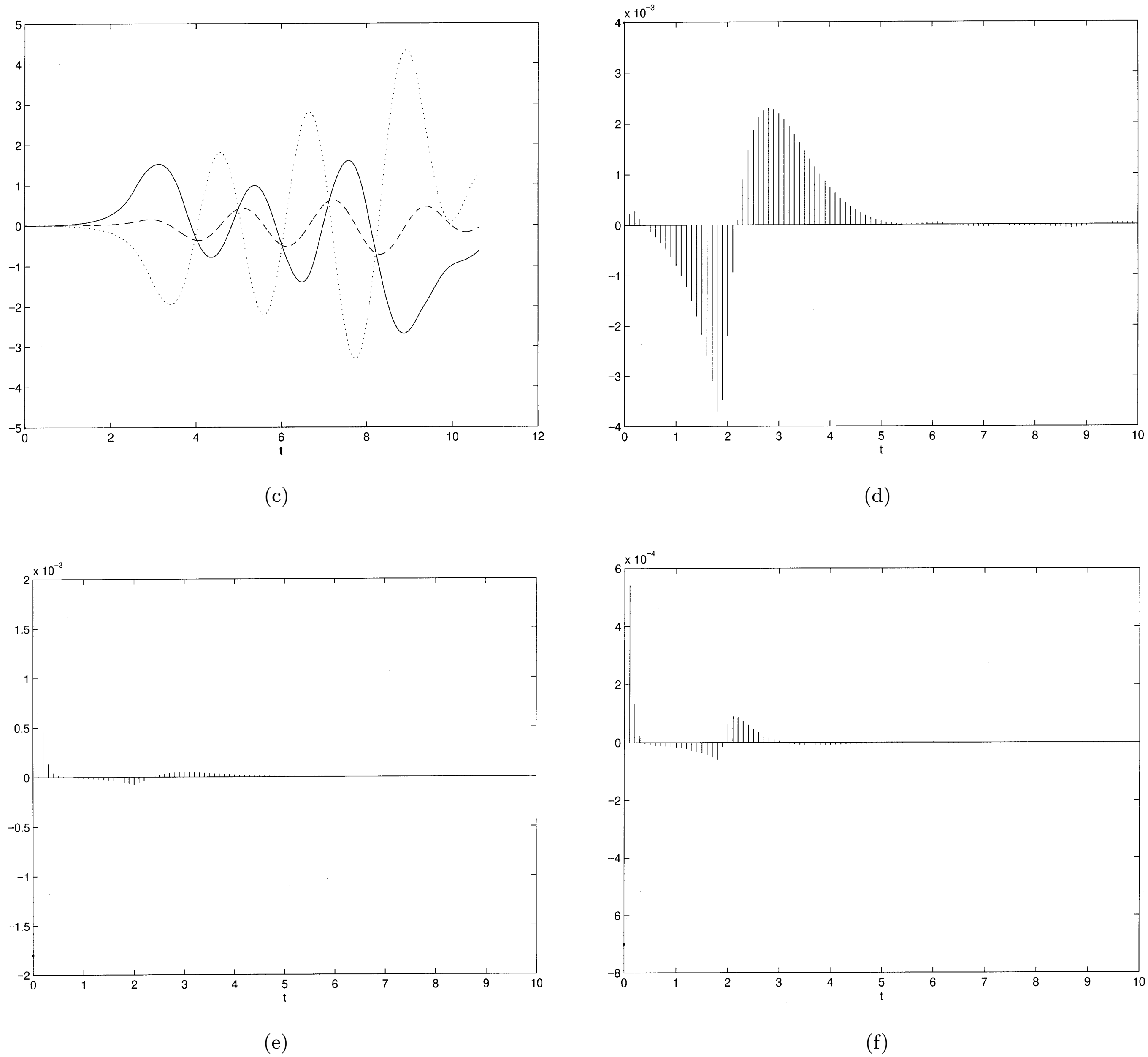


Fig. 4. (*Continued*)

output feedback and linear dynamic output feedback. Conditions for global asymptotic stability of the output error system are expressed as matrix inequalities. Simulation examples have been presented for Chua's circuit and coupled Chua's circuits that exhibit the double–double scroll attractor. In the latter case it was sufficient to measure one single variable and take one control input in order to obtain synchronization in the output. Often synchronization is also obtained for the full state vectors in addition to the theoretically guaranteed synchronization for the output vectors.

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