Robust Nonlinear H_∞ Synchronization of Chaotic Lur'e Systems

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***Abstract*— In this paper, we propose a method of robust nonlinear H_∞ master–slave synchronization for chaotic Lur'e systems with applications to secure communication. The scheme makes use of vector field modulation and either full static state or linear dynamic output error feedback control. The master–slave systems are assumed to be nonidentical and channel noise is taken into account. Binary valued continuous time message signals are recovered by minimizing the L_2-gain from the exogenous input to the tracking error for the standard plant representation of the scheme. The exogenous input takes into account the message signal, channel noise and parameter mismatch. Matrix inequality conditions for dissipativity with finite L_2-gain of the standard plant form are derived based on a quadratic storage function. The controllers are designed by solving a nonlinear optimization problem which takes into account both channel noise and parameter mismatch. The method is illustrated on Chua's circuit.**

***Index Terms*—Chua's circuit, Lur'e systems, matrix inequalities, parametric uncertainty, synchronization.**

I. INTRODUCTION

SECURE communication [3], [11] is an important field for the application of synchronization theory. The link between absolute stability theory and synchronization of nonlinear systems has been investigated in a series of papers [7], [8], [24], [33], in particular for Lur'e systems and master–slave synchronization schemes. From a control theoretic point of view, this corresponds to the autonomous case without an external input (or message signal). Among the methods that consider a message signal in the synchronization scheme, one basically makes a distinction between chaotic masking and vector field modulation (see Kennedy in [3]). With respect to vector field modulation, we have proposed a new method of nonlinear H_∞ synchronization [25], [26] which applies to binary valued continuous time message signals. The synchronization schemes are interpreted within the framework of modern control theory by taking standard plant representations. A new notion of synchronization error has been introduced which is based on the tracking error of the scheme. The aim of the nonlinear H_∞ synchronization scheme is to minimize the influence of the exogenous input on the regulated output. The exogenous input contains the message signal and channel noise. The design has been based on matrix inequalities which follow from conditions of dissipativity with finite L_2-gain of the synchronization scheme. Dissipativity of nonlinear systems is a well-known and fundamental system theoretical concept which dates back to the work of Willems and Hill and Moylan [12], [13], [32]. A difference between the method proposed in [25], [26] and methods of nonlinear H_∞ control theory such as [14], [30] is that in the former a quadratic storage function is chosen, while in the latter a general continuously differentiable nonlinear storage function is employed. In this way, matrix inequalities are obtained instead of a Hamilton–Jacobi inequality. The design of the controller has been achieved by solving a nonlinear optimization problem based on the matrix inequalities.

This previous work [25], [26] can be considered as a first step toward a *robust* synchronization theory. In this paper, we treat the problem of *parameter mismatch*, in addition to the problem of channel noise and take both into account in the controller design (adaptive control approaches to cope with parameter mismatch have been investigated, e.g., in [34]). We discuss the case of full static state error feedback and linear dynamic output error feedback. For identical master–slave systems this has been studied in [25], [26]. The class of nonlinear systems considered is in Lur'e form [16], [31]. Many systems of common interest such as Chua's circuit [4], [5], generalized Chua's circuits [28], arrays of such cells or cellular neural networks [6], [10], [15], [29] can be represented in Lur'e form. Chaotic or hyperchaotic behavior is obtained from these systems for the double scroll, n-scrolls, double-double scroll, and n-double scroll hypercube. For the autonomous case, i.e., without a message signal, parameter mismatch between the Lur'e systems has been investigated in [27]. One unexpected result of that study is that it is possible to allow a large parameter mismatch such that the systems remain master–slave synchronized up to a relatively small error. It has been illustrated on Chua's circuit in [27] that

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the chaotic slave system can be synchronized in this sense to a master system which behaves chaotically, shows limit cycle behavior or shows stable equilibrium behavior. Another example along this direction has been presented in [18], where full state error feedback has been used to synchronize two systems which are different (such as Chua's circuit and the Lorenz attractor). In this paper, we extend the ideas from the autonomous case to the case where there exists an external input. From the example of Chua's circuit it will follow that the allowed parameter mismatch is much smaller than for the autonomous case. By using a single transmission signal, the dynamic output feedback case leads to a simpler implementation of the synchronization than the full static state feedback scheme, but the latter has higher performance and a better flexibility for defining keys in a cryptographical scheme [25], [26].

This paper is organized as follows. In Section II, we present master–slave synchronization schemes with full static state error feedback and linear dynamic output error feedback. In Section III, we approach the synchronization problem from the viewpoint of modern control theory, by deriving standard plant representations. We take into account parameter mismatch between the Lur'e systems. In Section IV, we derive Theorems for dissipativity with finite L_2-gain of the synchronization schemes, these conditions being expressed as matrix inequalities. In Section V, we formulate the robust nonlinear H_∞ synchronization problem, based on the Theorems of Section IV. In Section VI, we present an example on Chua's circuit. Both static state and dynamic output feedback are applied and a comparison is made. Channel noise and parameter mismatch are taken into account in the design.

II. Synchronization Scheme

In this section, we consider the master–slave synchronization schemes with vector field modulation proposed in [25] and [26], but with parameter mismatch between the systems.

A. Full Static State Error Feedback

Consider the master–slave synchronization scheme with full static state error feedback for nonidentical master–slave Lur'e systems:

$$\begin{aligned}
\mathcal{R}&: \begin{cases} \dot{\mu} = R\mu + Sr \\ d = T\mu + Ur \end{cases} \\
\mathcal{M}_s&: \begin{cases} \dot{x} = A_1 x + B_1 \sigma(C_1 x) + Dd \\ p = H_s x \end{cases} \\
\mathcal{S}_s&: \begin{cases} \dot{z} = A_2 z + B_2 \sigma(C_2 z) + u_s \\ q = H_s z \end{cases} \\
\mathcal{C}_s&: u_s = F(p + \epsilon - q)
\end{aligned} \tag{1}$$

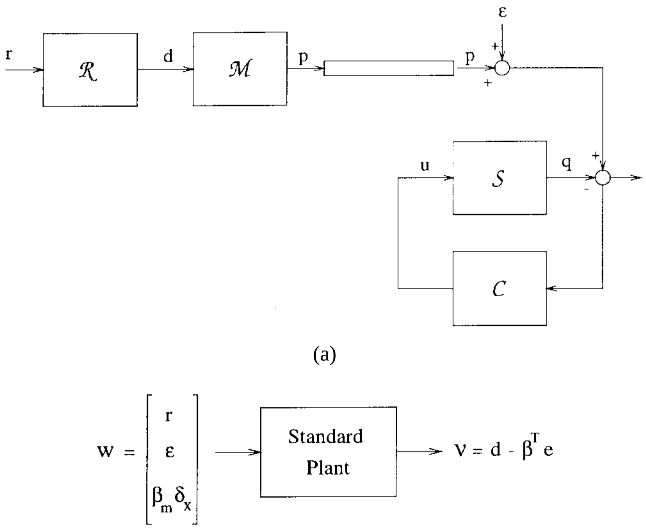
with master system $\mathcal{M}_s$, slave system $\mathcal{S}_s$, full static state error feedback controller $\mathcal{C}_s$, and linear filter $\mathcal{R}$ (Fig. 1). The index s refers to the *static* feedback case. The subsystems have state vectors $x, z \in \mathbb{R}^n$, $\mu \in \mathbb{R}^{n_r}$ and output vectors $p, q \in \mathbb{R}^n$, $u_s \in \mathbb{R}^n$, $d \in \mathbb{R}$. The message signal is $r \in \mathbb{R}$. At the transmitter $\mathcal{M}_s$, a linear transformation $H_s \in \mathbb{R}^{n\times n}$ is applied to the state vector x. The resulting vector p is sent along the channel and is corrupted by the disturbance signal or channel noise $\epsilon \in \mathbb{R}^n$. At the receiver, full static state error feedback between the output q of $\mathcal{S}_s$ and p is applied with feedback matrix $F \in \mathbb{R}^{n\times n}$. The nonidentical master–slave Lur'e systems have system matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, $B_1, B_2 \in \mathbb{R}^{n\times n_h}$, and $C_1, C_2 \in \mathbb{R}^{n_h\times n}$, where n_h corresponds to the number of hidden units (if one interprets the Lur'e system as a class of recurrent neural networks [16], [23], [31]). The diagonal nonlinearity $\sigma(.)$: $\mathbb{R}^{n_h} \mapsto \mathbb{R}^{n_h}$ is assumed to belong to sector $[0, k]$ [16], [31]. At the master system the vector field is modulated by means of the term Dd with $D \in \mathbb{R}^{n\times 1}$. We choose message signals r which satisfy $\|r\|_\infty = \sup_{t\geq 0}|r(t)| \leq 1$ and are binary valued. As a typical test signal for the synchronization scheme, signals of the form sign $(\cos \omega t)$ will be employed. When taking a chaotic Lur'e system, the norm of D is chosen "small" (compared to the norm of the other terms in the system dynamics) in order to hide the message signal in the strange attractor. Furthermore, we assume that the master system possesses an initial state such that it is input to state stable for the considered class of message signals (see Assumption 2 in the sequel). The low pass filter $\mathcal{R}$ has system matrices $R \in \mathbb{R}^{n_r\times n_r}$, $S \in \mathbb{R}^{n_r\times 1}$, $T \in \mathbb{R}^{1\times n_r}$, $U \in \mathbb{R}$. In the synchronization scheme, the original message will be recovered from one of the components of the signal $p - q$.

B. Dynamic Output Error Feedback

Consider now the master–slave synchronization scheme with dynamic output error feedback and nonidentical master–slave Lur'e systems:

$$\begin{aligned}
\mathcal{R}&: \begin{cases} \dot{\mu} = R\mu + Sr \\ d = T\mu + Ur \end{cases} \\
\mathcal{M}_d&: \begin{cases} \dot{x} = A_1 x + B_1 \sigma(C_1 x) + Dd \\ p = H_d x \end{cases} \\
\mathcal{S}_d&: \begin{cases} \dot{z} = A_2 z + B_2 \sigma(C_2 z) + F u_d \\ q = H_d z \end{cases} \\
\mathcal{C}_d&: \begin{cases} \dot{\rho} = E\rho + G(p + \epsilon - q) \\ u_d = M\rho + N(p + \epsilon - q) \end{cases}
\end{aligned} \tag{2}$$

with master system $\mathcal{M}_d$, slave system $\mathcal{S}_d$, linear dynamic output feedback controller $\mathcal{C}_d$, and linear filter $\mathcal{R}$ (Fig. 1). The index d refers to the *dynamic* feedback case. The subsystems have state vectors $x, z \in \mathbb{R}^n$, $\mu \in \mathbb{R}^{n_r}$, $\rho \in \mathbb{R}^{n_c}$, and output vectors $p, q \in \mathbb{R}^l$, $u_d \in \mathbb{R}^m$, $d \in \mathbb{R}$, where $l, m \leq n$. The message signal is $r \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^l$ is a disturbance input. At the transmitter $\mathcal{M}_d$, a linear transformation $H_d \in \mathbb{R}^{l\times n}$ is applied to the state vector x. The resulting vector p is transmitted along the channel. At the receiver $\mathcal{S}_d$, linear dynamic output error feedback is applied by taking the difference between p and q as input to the



(a)

(b)

Fig. 1. (a) Synchronization scheme with master system $\mathcal{M}$ and slave system $\mathcal{S}$. Vector field modulation is applied to $\mathcal{M}$ by means of the signal d, which is the output of the low pass filter $\mathcal{R}$ with as input the message signal r. For $\mathcal{M}$ and $\mathcal{S}$ we consider Lur'e systems with parameter mismatch between the systems. The outputs of $\mathcal{M}$ and $\mathcal{S}$ are p and q, which are linear transformations of the state variables x and z, respectively. The signal p is sent along the channel and is corrupted by means of the signal ϵ. A binary valued continuous time message signal is considered which is recovered by defining a tracking error for the overall system and applying a controller $\mathcal{C}$ to the slave system. We consider the cases of full static state error feedback and linear dynamic output feedback for this controller, leading to the schemes $\{\mathcal{R}, \mathcal{M}_s, \mathcal{S}_s, \mathcal{C}_s\}$ and $\{\mathcal{R}, \mathcal{M}_d, \mathcal{S}_d, \mathcal{C}_d\}$, respectively. (b) Control theoretic interpretation of the synchronization scheme by means of its standard plant representation with exogenous input w and regulated output ν. The aim of *robust nonlinear* H_∞ *synchronization* is to minimize the influence from the exogenous input on the regulated output. The exogenous input contains the message signal, the disturbance signal ϵ and the parameter $\beta_m \delta_x$ related to the parameter mismatch between the master–slave systems.

controller with system matrices $E \in \mathbb{R}^{n_c \times n_c}$, $G \in \mathbb{R}^{n_c \times l}$, $M \in \mathbb{R}^{m \times n_c}$, $N \in \mathbb{R}^{m \times l}$. Furthermore, $F \in \mathbb{R}^{n \times m}$. The transmitted signal p is corrupted by the signal ϵ. The system matrices of the master–slave Lur'e systems, the nonlinearity $\sigma(.)$, the vector field modulation and the low-pass filter $\mathcal{R}$ are the same as for the scheme (1). The same class of message signals is considered as in the state feedback case, but will be recovered from one of the components of the signal $x - z$.

III. Standard Plant Representations

In this section, we derive standard plant representations for the synchronization schemes (1) and (2), taking into account the parameter mismatch between the master–slave systems.

A. Full Static State Error Feedback

Defining $e_s = p - q$ and denoting the state equation of synchronization scheme (1) as

$$\begin{cases} \dot{x} = f_s(x, \mu, r) \\ \dot{z} = g_s(z, x, \epsilon) \end{cases} \tag{3}$$

with continuous nonlinear mappings $f_s(., ., .)$: $\mathbb{R}^n \times \mathbb{R}^{n_r} \times \mathbb{R} \mapsto \mathbb{R}^n$ and $g_s(., ., .)$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$, one obtains

$$\dot{e}_s = H_s[f_s(x, \mu, r) - g_s(z, x, \epsilon)]. \tag{4}$$

According to the proof of Theorem 14 in [27] and [33], we decompose this as

$$\dot{e}_s = H_s[v_s(x, z, \epsilon) + w_s(x, \mu, \epsilon)] \tag{5}$$

with

$$\begin{cases} v_s(x, z, \epsilon) = g_s(x, x, \epsilon) - g_s(z, x, \epsilon) \\ \qquad\qquad = (A_2 H_s^{-1} - F)e_s + B_2 \eta(C_2 H_s^{-1} e_s; z) \\ w_s(x, \mu, \epsilon) = f_s(x, \mu, r) - g_s(x, x, \epsilon) \\ \qquad\qquad = DT\mu + DUr - F\epsilon + \varphi(x) \end{cases}$$

with $\varphi(x) = (A_1 - A_2)x + B_1\sigma(C_1 x) - B_2\sigma(C_2 x)$ and $\eta(C_2 H_s^{-1} e_s; z) = \sigma(C_2 H_s^{-1} e_s + C_2 z) - \sigma(C_2 z)$. According to [25], we define the tracking error $\nu = d - \beta^T e_s$, where $\beta = [1; 0; 0; \cdots; 0]$ selects the first component of e_s. The main motivation for defining this tracking error is that the signal e_s cannot converge to zero when an external input is applied to the master system. For the synchronization scheme, we obtain then the standard plant representation [1], [19] (Fig. 1)

$$\begin{cases} \begin{bmatrix} \dot{e}_s \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} A_{2_*} & H_s DT \\ 0 & R \end{bmatrix} \begin{bmatrix} e_s \\ \mu \end{bmatrix} + \begin{bmatrix} B_{2_*} \\ 0 \end{bmatrix} \eta(C_{2_*} e_s; z) \\ \qquad + \begin{bmatrix} H_s DU & -H_s F & H_s \\ S & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ \epsilon \\ \varphi(x) \end{bmatrix} \\ \nu = [-\beta^T \quad T] \begin{bmatrix} e_s \\ \mu \end{bmatrix} + [U \quad 0 \quad 0] \begin{bmatrix} r \\ \epsilon \\ \varphi(x) \end{bmatrix} \end{cases} \tag{6}$$

with state vector $\xi_s = [e_s; \mu]$ and regulated output ν. The interpretation for the exogenous input will be given in Section IV. By definition, one has $A_{2_*} = H_s A_2 H_s^{-1} - H_s F$, $B_{2_*} = H_s B_2$, $C_{2_*} = C_2 H_s^{-1}$. Note that the system matrices of the standard plant representation do not depend on A_1, B_1, C_1. The influence of the parameter mismatch is contained in $\varphi(x)$.

B. Dynamic Output Error Feedback

Defining $e_d = x - z$ and denoting the state equation of synchronization scheme (2) as

$$\begin{cases} \dot{x} = f_d(x, \mu, r) \\ \dot{z} = g_d(z, x, \rho, \epsilon) \\ \dot{\rho} = h_d(\rho, x, z, \epsilon) \end{cases} \tag{7}$$

with continuous nonlinear mappings $f_d(., ., .)$: $\mathbb{R}^n \times \mathbb{R}^{n_r} \times \mathbb{R} \mapsto \mathbb{R}^n$, $g_d(., ., ., .)$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_c} \times \mathbb{R}^l \mapsto \mathbb{R}^n$, and $h_d(., ., ., .)$: $\mathbb{R}^{n_c} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l \mapsto \mathbb{R}^{n_c}$, one obtains

$$\dot{e}_d = f_d(x, \mu, r) - g_d(z, x, \rho, \epsilon). \tag{8}$$

Like in the state feedback case, we decompose this as

$$\dot{e}_d = v_d(x, z, \rho, \epsilon) + w_d(x, \mu, \rho, \epsilon) \tag{9}$$

where

$$\begin{cases} v_d(x, z, \rho, \epsilon) = g_d(x, x, \rho, \epsilon) - g_d(z, x, \rho, \epsilon) \\ \qquad = (A_2 - FNH_d)e_d + B_2\eta(C_2 e_d; z) \\ w_d(x, \mu, \rho, \epsilon) = f_d(x, \mu, r) - g_d(x, x, \rho, \epsilon) \\ \qquad = DT\mu - FM\rho + DUr - FN\epsilon + \varphi(x) \end{cases}$$

with $\eta(C_2 e_d; z) = \sigma(C_2 e_d + C_2 z) - \sigma(C_2 z)$ and $\varphi(x)$ as defined in the state error feedback case. Defining the tracking error $\nu = d - \beta^T e_d$, the following standard plant representation is obtained (Fig. 1):

$$\begin{cases} \begin{bmatrix} \dot{e}_d \\ \dot{\rho} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} A_2 - FNH_d & -FM & DT \\ GH_d & E & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} e_d \\ \rho \\ \mu \end{bmatrix} \\ \qquad + \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix} \eta(C_2 e_d; z) + \begin{bmatrix} DU & -FN & I \\ 0 & G & 0 \\ S & 0 & 0 \end{bmatrix} \\ \qquad \cdot \begin{bmatrix} r \\ \epsilon \\ \varphi(x) \end{bmatrix} \\ \nu = [-\beta^T \quad 0 \quad T] \begin{bmatrix} e_d \\ \rho \\ \mu \end{bmatrix} + [U \quad 0 \quad 0] \begin{bmatrix} r \\ \epsilon \\ \varphi(x) \end{bmatrix} \end{cases} \tag{10}$$

with state vector $\xi_d = [e_d; \rho; \mu]$ and regulated output ν. For the interpretation of the exogenous input we refer again to Section IV.

IV. Dissipativity With Finite L_2-Gain

In this section, we first formulate assumptions on the nonlinearity and the boundedness of the trajectories of the master system. An interpretation for the exogenous input of the standard plant representations is given. Then conditions for dissipativity with finite L_2-gain and a quadratic storage function are derived for the synchronization schemes. These conditions are expressed as matrix inequalities.

We make the following two assumptions.

Assumption 1: The nonlinearity $\eta(C_2 e_d; z)$ in (10) belongs to sector $[0, k]$:

$$0 \leq \frac{\eta_i(c_{2_i}^T e_d; z)}{c_{2_i}^T e_d} = \frac{\sigma_i(c_{2_i}^T e_d + c_{2_i}^T z) - \sigma_i(c_{2_i}^T z)}{c_{2_i}^T e_d} \leq k, \qquad \forall e_d, z; \; i = 1, \cdots, n_h \, (c_{2_i}^T e_d \neq 0) \tag{11}$$

where $c_{2_i}^T$ denotes the ith row vector of C_2. The same assumption is made for $\eta(C_{2_*} e_s; z)$ in (6).

The following inequalities hold [2], [16], [31]:

$$\begin{aligned} &\eta_i(c_{2*_i}^T e_s; z)\,[\eta_i(c_{2*_i}^T e_s; z) - k c_{2*_i}^T e_s] \\ &\quad \leq 0, \qquad \forall e_s, z; \qquad i = 1, \cdots, n_h \\ &\eta_i(c_{2_i}^T e_d; z)\,[\eta_i(c_{2_i}^T e_d; z) - k c_{2_i}^T e_d] \\ &\quad \leq 0, \qquad \forall e_d, z; \qquad i = 1, \cdots, n_h. \end{aligned} \tag{12}$$

It follows from the mean value theorem that for differentiable $\sigma(.)$ the sector condition $[0, k]$ on $\eta(.)$ corresponds to [7]

$$0 \leq \frac{d}{d\rho}\sigma_i(\rho; z) \leq k, \qquad \forall \rho, z; \qquad i = 1, \cdots, n_h. \tag{13}$$

Assumption 2: The master systems $\mathcal{M}_s$ and $\mathcal{M}_d$ are input to state stable in the sense that there exist initial states x_0 and a positive real constant δ_x such that

$$\|x(t)\|_2 \leq \delta_x, \qquad \forall t \in [0, \infty) \tag{14}$$

for all continuous time reference inputs r which satisfy $\|r\|_\infty = \sup_{t \geq 0} |r(t)| \leq 1$.

The viewpoint that we take here is pragmatic in the sense that in practice one is not interested in employing a master system that possesses unbounded trajectories. Note that for a zero external input d the upper bound δ_x is a measure for the "size" of the attractor of the chaotic master system [7]. The Lur'e systems, matrix D and initial states are chosen such that the master system satisfies Assumption 2. Using the expression for $\varphi(x)$ one obtains $\|\varphi(x)\|_2 \leq \beta_m \|x\|_2 \; \forall x$ with $\beta_m = \|\Delta A\|_2 + k\|B_1\|_2 \|C_1\|_2 + k\|B_2\|_2 \|C_2\|_2$ and $\Delta A = A_1 - A_2$. In case $C_1 = C_2$ one obtains $\beta_m = \|\Delta A|_2 + k\|\Delta B\|_2 \|C_2\|_2$ where $\Delta B = B_1 - B_2$. From Assumption 2, one has $\|\varphi(x)\|_2 < \beta_m \delta_x, \; \forall x$. Note that this upper bound might be conservative. On the other hand, this approach has led to useful criteria for robust synchronization of the autonomous synchronization scheme, discussed in [27]. For the upper bound we will consider a positive constant scaling factor α

$$\|\varphi(x)\|_2 \leq \alpha \beta_m \delta_x. \tag{15}$$

In order to analyze the I/O properties of the standard plant representation of the synchronization scheme with static state feedback (6), we consider the quadratic storage function [12], [13], [32]:

$$\phi(\xi_s) = \xi_s^T P \xi_s = [e_s^T \quad \mu^T] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} e_s \\ \mu \end{bmatrix}, \quad P = P^T > 0 \tag{16}$$

and a supply rate with finite L_2-gain γ:

$$s(w, \nu) = \gamma^2 w^T w - \nu^T \nu \tag{17}$$

with regulated output ν and exogenous input w. As exogenous input we take $w = [r; \epsilon; \beta_m \delta_x]$, which consists of the reference input, disturbance signals and a constant signal related to the parameter mismatch between the master–slave systems. The system (6) is said to be dissipative [12], [13], [32] with respect to supply rate (17) and the storage function (16) if $\dot{\phi} \leq s(w, \nu), \; \forall w, \nu$. The following Theorem holds.

Theorem 1: Let $\Lambda = \text{diag}\,\{\lambda_i\}$ be a diagonal matrix with $\lambda_i \geq 0$ for $i = 1, \cdots, n_h$. A sufficient condition for dissipativity of the synchronization scheme with full static state feedback (6) with respect to the storage function (16) and the supply rate with L_2-gain γ (17) is given by the matrix inequality

$$Z = Z^T = \left[\begin{array}{ccccc|c} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & P_{11}H \\ \cdot & Z_{22} & Z_{23} & Z_{24} & Z_{25} & P_{21}H \\ \cdot & \cdot & Z_{33} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & Z_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & Z_{55} & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & -\dfrac{\gamma^2 I}{\alpha^2} \end{array}\right] < 0 \quad (18)$$

with

$$\begin{aligned} Z_{11} &= A_{2_*}^T P_{11} + P_{11} A_{2_*} + \beta\beta^T \\ Z_{12} &= A_{2_*}^T P_{12} + P_{11} H_s D T + P_{12} R - \beta T \\ Z_{13} &= P_{11} B_{2_*} + k C_{2_*}^T \Lambda \\ Z_{14} &= P_{11} H_s D U + P_{12} S - \beta U \\ Z_{15} &= -P_{11} H_s F \\ Z_{22} &= P_{21} H_s D T + T^T D^T H_s^T P_{12} \\ &\quad + P_{22} R + R^T P_{22} + T^T T \\ Z_{23} &= P_{21} B_{2_*} \\ Z_{24} &= P_{21} H_s D U + P_{22} S + T^T U \\ Z_{25} &= -P_{21} H_s F \\ Z_{33} &= -2\Lambda, \\ Z_{44} &= -\gamma^2 I + U^T U, \\ Z_{55} &= -\gamma^2 I. \end{aligned}$$

Proof: The condition (15) is expressed as $\beta_m^2 \delta_x^2 - \varphi^T \varphi / \alpha^2 \geq 0$. Together with the sector condition on η (11), this condition is employed in an application of the $\mathcal{S}$-procedure ([2, p. 23]) in checking the condition $\dot{\phi} - s \leq 0$. This means that positive real constants λ_i and τ are introduced such that

$$\dot{\phi} - s - 2 \sum_i \lambda_i \eta_i (\eta_i - k c_{2_*}^T e_s) + \tau \left(\beta_m^2 \delta_x^2 - \frac{\varphi^T \varphi}{\alpha^2} \right) < 0$$

and by defining $\Lambda = \operatorname{diag}\{\lambda_i\}$:

$$\dot{\phi} - s - 2\eta^T \Lambda (\eta - k C_{2_*} e_s) + \tau \left(\beta_m^2 \delta_x^2 - \frac{\varphi^T \varphi}{\alpha^2} \right) < 0.$$

Using the expression for the supply rate $s = \gamma^2 (r^T r + \epsilon^T \epsilon + \beta_m^2 \delta_x^2) - \nu^T \nu$ and choosing $\tau = \gamma^2$ this condition becomes

$$\begin{aligned} &\dot{\phi} - \gamma^2 (r^T r + \epsilon^T \epsilon) + \nu^T \nu - 2\eta^T \Lambda (\eta - k C_{2_*} e_s) \\ &\quad - \frac{\gamma^2 \varphi^T \varphi}{\alpha^2} < 0. \end{aligned}$$

Using $\dot{\phi} = \dot{\xi}_s^T P \xi_s + \xi_s^T P \dot{\xi}_s$ and the equation of the standard plant representation (6), this can be written as the quadratic form $\zeta^T Z \zeta < 0$ with $\zeta = [e_s; \mu; \eta; r; \epsilon; \varphi]$. This expression is negative for all nonzero ζ if Z is negative definite. □

In order to analyze the I/O properties of the standard plant representation of the synchronization scheme with dynamic output feedback (10), we consider the quadratic storage function

$$\phi(\xi_d) = \xi_d^T P \xi_d = [e_d^T \quad \rho^T \quad \mu^T] \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} e_d \\ \rho \\ \mu \end{bmatrix}$$
$$P = P^T > 0 \quad (19)$$

and the supply rate with finite L_2-gain γ (17) and exogenous input $w = [r; \epsilon; \beta\delta_x]$.

Theorem 2: Let $\Lambda = \operatorname{diag}\{\lambda_i\}$ be a diagonal matrix with $\lambda_i \geq 0$ for $i = 1, \cdots, n_h$. A sufficient condition for dissipativity of the synchronization scheme with dynamic output feedback (10) with respect to the storage function (19) and the supply rate with L_2-gain γ (17) is given by the matrix inequality

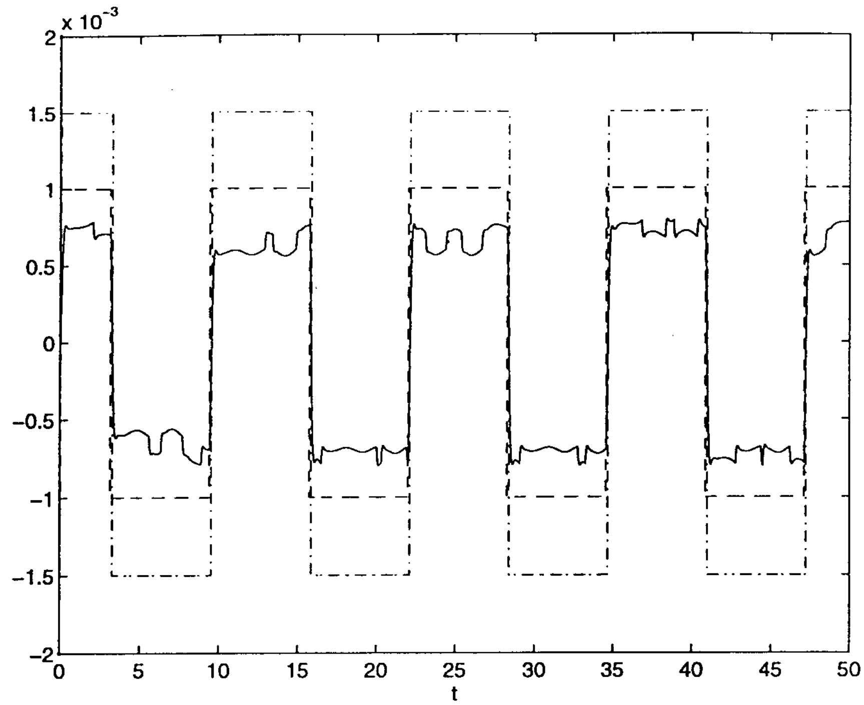
$$Z = Z^T = \left[\begin{array}{cccccc|c} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & P_{11} \\ \cdot & Z_{22} & Z_{23} & Z_{24} & Z_{25} & Z_{26} & P_{21} \\ \cdot & \cdot & Z_{33} & Z_{34} & Z_{35} & Z_{36} & P_{31} \\ \cdot & \cdot & \cdot & Z_{44} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & Z_{55} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & Z_{66} & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dfrac{-\gamma^2 I}{\alpha^2} \end{array}\right] < 0 \quad (20)$$

with

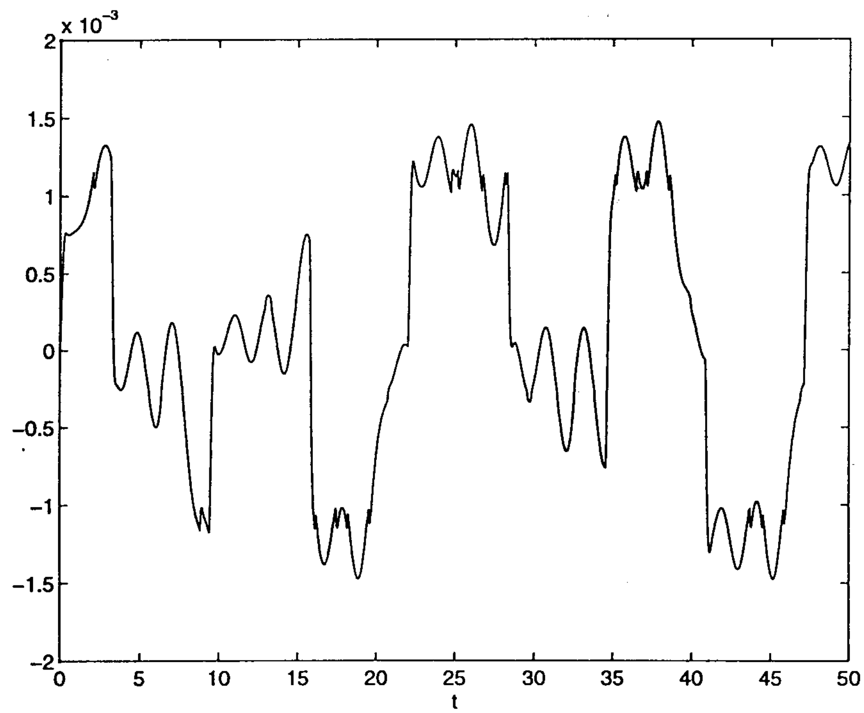
$$\begin{aligned} Z_{11} &= (A_2 - FNH_d)^T P_{11} + P_{11}(A_2 - FNH_d) \\ &\quad + H_d^T G^T P_{21} + P_{12} G H_d + \beta\beta^T \\ Z_{12} &= (A_2 - FNH_d)^T P_{12} + H_d^T G^T P_{22} - P_{11} F M + P_{12} E \\ Z_{13} &= (A_2 - FNH_d)^T P_{13} + H_d^T G^T P_{23} \\ &\quad + P_{11} D T + P_{13} R - \beta T \\ Z_{14} &= P_{11} B_2 + k C_2^T \Lambda \\ Z_{15} &= P_{11} D U + P_{13} S - \beta U \\ Z_{16} &= P_{12} G - P_{11} F N \\ Z_{22} &= E^T P_{22} + P_{22} E - M^T F^T P_{12} - P_{21} F M \\ Z_{23} &= E^T P_{23} - M^T F^T P_{13} + P_{21} D T + P_{23} R \\ Z_{24} &= P_{21} B_2 \\ Z_{25} &= P_{21} D U + P_{23} S \\ Z_{26} &= P_{22} G - P_{21} F N \\ Z_{33} &= R^T P_{33} + P_{33} R + T^T D^T P_{13} + P_{31} D T + T^T T \\ Z_{34} &= P_{31} B_2 \\ Z_{35} &= P_{31} D U + P_{33} S + T^T U \\ Z_{36} &= P_{32} G - P_{31} F N \\ Z_{44} &= -2\Lambda \\ Z_{55} &= -\gamma^2 I + U^T U \\ Z_{66} &= -\gamma^2 I. \end{aligned}$$

Proof: According to the proof of Theorem 1, we use the inequality $\beta_m^2 \delta_x^2 - \varphi^T \varphi / \alpha^2 \geq 0$ and the inequality from the sector condition on η. By employing these inequalities in the $\mathcal{S}$-procedure, one obtains

$$\begin{aligned} &\dot{\phi} - \gamma^2 (r^T r + \epsilon^T \epsilon) + \nu^T \nu - 2 \sum_i \lambda_i \eta_i (\eta_i - k c_2^T e_d) \\ &\quad - \frac{\gamma^2 \varphi^T \varphi}{\alpha^2} < 0 \end{aligned}$$



(a)



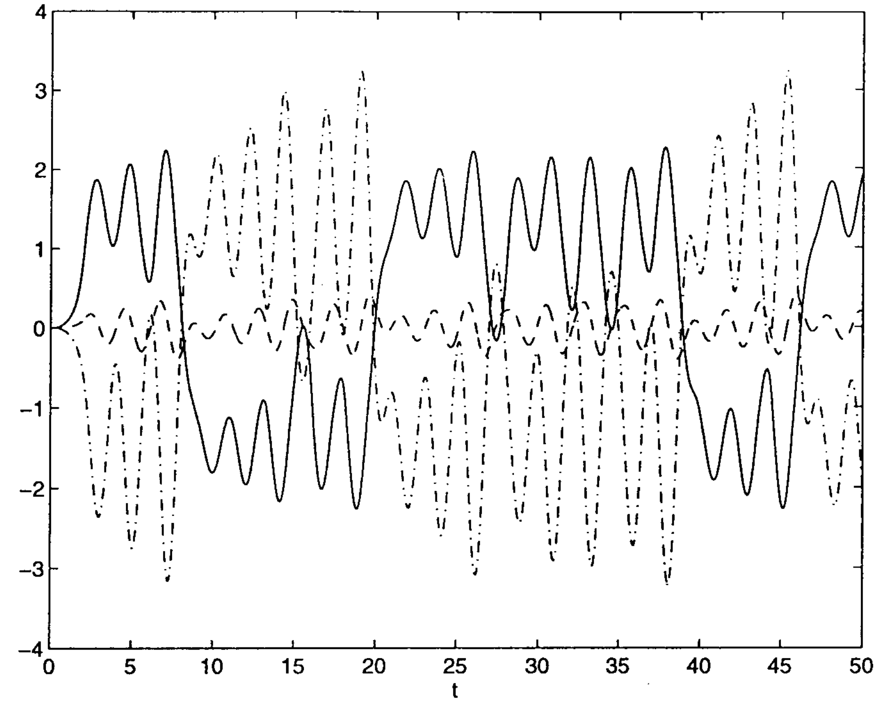
(b)

Fig. 2. Robust nonlinear H_∞ synchronization of Chua's circuit using full static state error feedback. (a) $\beta^T e_s$ (-), scaled version of message signal sign $[\cos(0.5t)]$ (- -) and scaled version of recovered message signal $\mathrm{sign}(\beta^T e_s)$ (-.). The parameter mismatch of the master with respect to the slave system is $\delta a_{11} = 0.001$. (b) $\beta^T e_s$ for a too large parameter mismatch $\delta a_{11} = 0.01$. The original message is not recovered in this case.

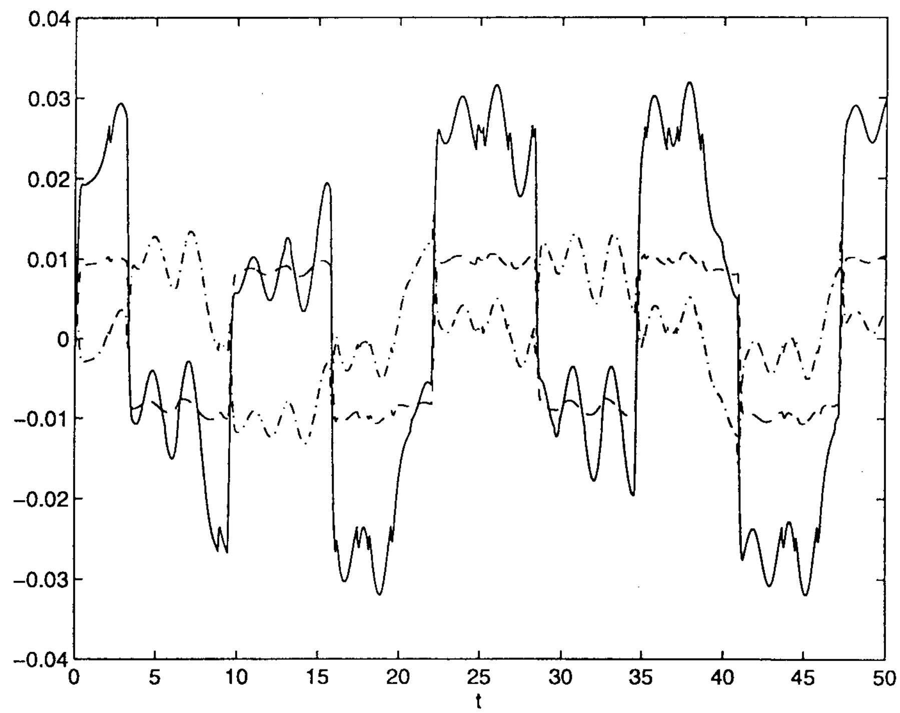
and by defining $\Lambda = \mathrm{diag}\,\{\lambda_i\}$:

$$\dot{\phi} - \gamma^2(r^T r + \epsilon^T \epsilon) + \nu^T \nu - 2\eta^T \Lambda(\eta - kC_2 e_d) - \frac{\gamma^2 \varphi^T \varphi}{\alpha^2} < 0.$$

Using $\dot{\phi} = \dot{\xi}_d^T P \xi_d + \xi_d^T P \dot{\xi}_d$ and the equation of the standard representation (10) this can be written as the quadratic form $\zeta^T Z \zeta < 0$, with $\zeta = [e_d; \rho; \mu; \eta; r; \epsilon; \varphi]$. This expression is negative for all nonzero ζ if Z is negative definite. □



(a)



(b)

Fig. 3. Static state error feedback (continued). (a) Transmitted signals p which are a linear transformation of the state variables x of the master system, with p_1 (-), p_2 (- -), p_3 (-.). Applying vector field modulation, the message signal is invisible on the chaotic carrier signal. (b) Control signal u applied to the slave system using full static state error feedback, with u_1 (-), u_2 (- -), u_3 (-.).

V. Robust Nonlinear H_∞ Synchronization

In this section, we explain how to design the controllers $\mathcal{C}_s$ and $\mathcal{C}_d$ based on the matrix inequalities (18) and (20). In nonlinear H_∞ control theory (see e.g., [14], [30]) a controller for a given nonlinear plant is designed by considering a supply rate with finite L_2-gain. The optimal nonlinear H_∞ control law corresponds to the minimal achievable L_2-gain which makes the closed-loop system dissipative. The optimal solution is characterized by means of a Hamilton–Jacobi inequality with respect to a general continuously differentiable nonlinear

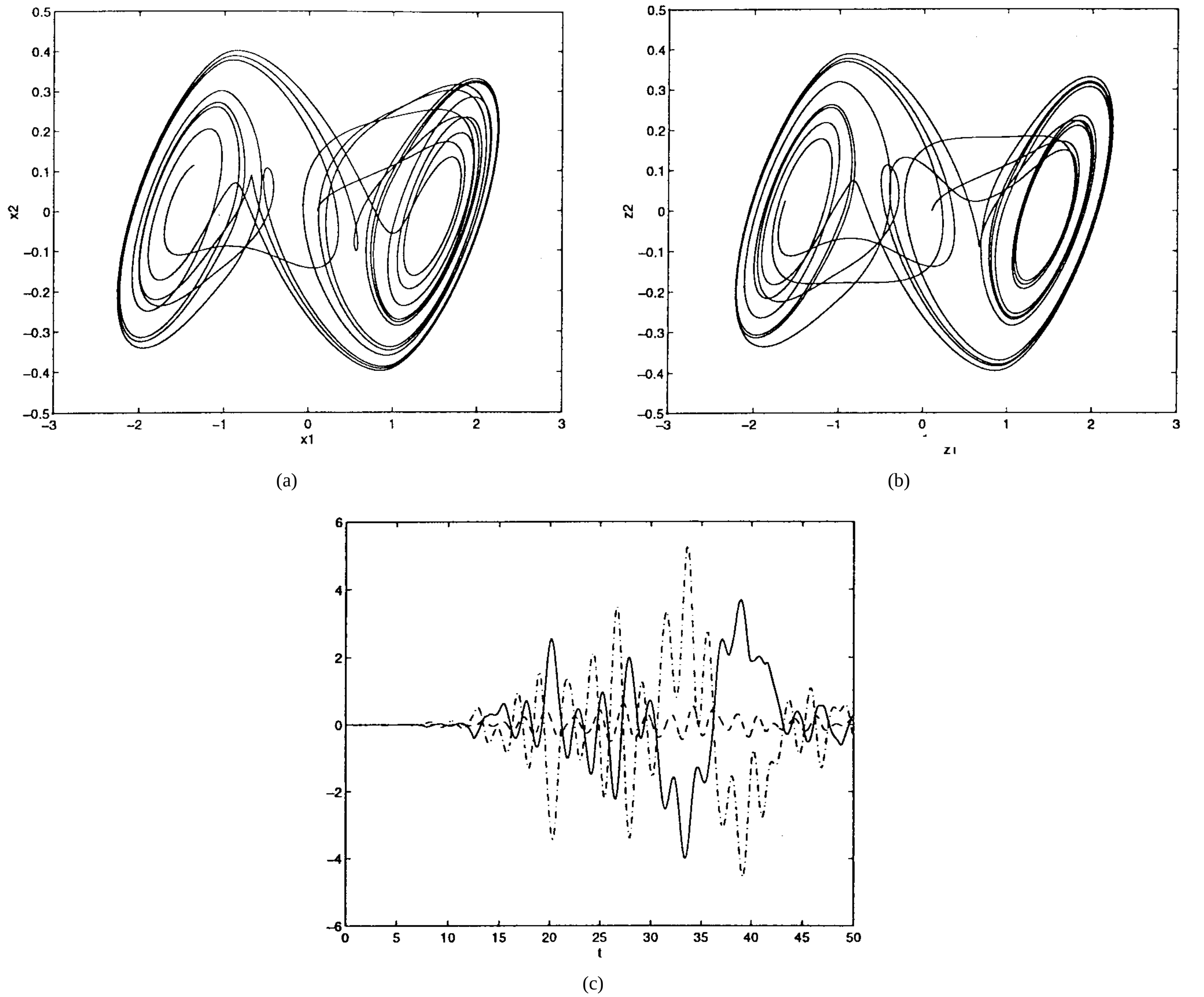


Fig. 4. Static state error feedback (continued). Behavior of the master and slave systems in the autonomous case, for the parameter mismatch of Fig. 2(a) and initial condition $x(0) = z(0) = [0.1;\, 0;\, 0]$. (a) $(x_1,\, x_2)$ of the master system. (b) $(z_1,\, z_2)$ of the slave system. (c) $x - z$ with respect to time.

storage function. On the other hand, in our previous work on nonlinear H_∞ synchronization and in the present paper, we consider a quadratic storage function which leads to matrix inequalities. The nonlinear H_∞ synchronization problem, as defined in [25], [26], corresponds to

$$\min_{\theta_c,\, P,\, \Lambda,\, \gamma} \gamma \text{ such that } \begin{cases} Z(\theta_c,\, P,\, \Lambda,\, \gamma,\, \alpha) < 0 \\ P = P^T > 0,\, \Lambda \geq 0 \text{ and diagonal} \end{cases} \tag{21}$$

where θ_c denotes the parameter vector of the controller $\mathcal{C}_s$ or $\mathcal{C}_d$, i.e., $\theta_c = F(:)$ or $\theta_c = [E(:);\, G(:);\, M(:);\, N(:)]$, respectively, where "(:)" denotes a columnwise scan of a matrix. In the *robust* nonlinear H_∞ synchronization problem, the parameter α is maximized, in order to achieve maximal robustness with respect to parameter mismatch between the master–slave systems, as follows from (15). Using a penalty method [9] the problem can be formulated as follows:

$$\min_{\theta_c, P, \Lambda, \gamma, \alpha} \gamma + c\,\frac{1}{\alpha} \text{ such that } \begin{cases} Z(\theta_c,\, P,\, \Lambda,\, \gamma,\, \alpha) < 0 \\ P = P^T > 0,\, \Lambda \geq 0 \end{cases} \tag{22}$$

where c is a positive real constant. In this way, the influence from the exogenous input on the regulated output is minimized, taking into account the reference input (message signal), the disturbance signal ϵ and the parameter mismatch. Because it is well-known from control theory that perfect tracking is impossible for all possible reference input signal, binary valued continuous time reference inputs r are considered such that the message signal can be recovered from sign $(\beta^T e_{s,\,d})$ [25], [26]. The constraint $P > 0$ can be eliminated by taking the parametrization $P = Q^T Q$. The same applies to Λ. In practice, one solves

$$\min_{\theta_c,\, Q,\, \Lambda,\, \gamma,\, \alpha} \gamma + c\,\frac{1}{\alpha} \text{ such that}$$
$$\lambda_{\max}[Z(\theta_c,\, Q,\, \Lambda,\, \gamma,\, \alpha)] + \delta < 0 \tag{23}$$

where $\lambda_{\max}[.]$ denotes the maximal eigenvalue of a symmetric matrix and δ a small positive constant. The constraint is differentiable as long as the two largest eigenvalues of Z do not coincide. Otherwise a generalized gradient can be defined [22].

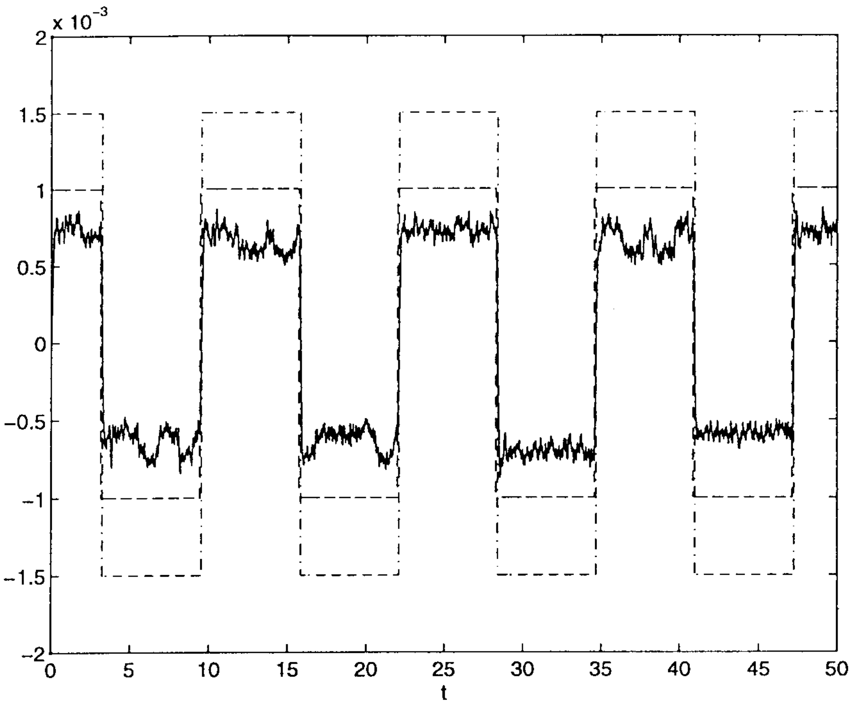


Fig. 5. Static state error feedback (continued). Simulation of the synchronization scheme for the same case as Fig. 2(a) but with zero mean white Gaussian channel noise ϵ with standard deviation 0.0001.

The optimization problem is nonconvex. However, suboptimal solutions yield satisfactory results as we will show by an example in the following section.

VI. Example: Chua's Circuit

In this section, we illustrate the robust nonlinear H_∞ synchronization method on Chua's circuit:

$$\begin{cases} \dot{x}_1 = a[x_2 - h(x_1)] \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -bx_2 \end{cases} \quad (24)$$

with nonlinear characteristic

$$h(x_1) = m_1 x_1 + \tfrac{1}{2}(m_0 - m_1)(|x_1 + 1| - |x_1 - 1|)$$

and parameters $a = 9$, $b = 14.286$, $m_0 = -\frac{1}{7}$, $m_1 = \frac{2}{7}$ in order to obtain the double scroll attractor [4], [5], [20]. The nonlinearity $\sigma(x_1) = \frac{1}{2}(|x_1 + 1| - |x_1 - 1|)$ (linear characteristic with saturation) belongs to sector [0, 1]. Hence, Chua's circuit can be interpreted as a Lur'e system $\dot{x} = Ax + B\sigma(Cx)$ where

$$A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, B = \begin{bmatrix} -a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix}$$
$$C = [1 \quad 0 \quad 0]. \quad (25)$$

We assign the double scroll behavior to the slave system by taking $A_2 = A$, $B_2 = B$, $C_2 = C$. The parameter mismatch is considered with respect to this nominal slave system.

We first consider the synchronization scheme with full static state error feedback (1). Cryptographical aspects of this scheme are discussed in [25], where H_s or an additional multilayer perceptron with square and full rank interconnection matrices may be used for the definition of a secret key, used by sender and receiver for enciphering and deciphering. We illustrate the working of the scheme here for $D = [1; 1; 1]$, $H_s = I$ and $\beta = [1; 0; 0]$. For the reference model $\mathcal{R}$ a first order Butterworth filter is chosen with cut-off frequency 10 Hz. For robust nonlinear H_∞ synchronization, the nonlinear optimization problem (23) has been solved with $c = 1$, $\delta = 0.01$. In order to limit the control energy, an additional constraint $\|\theta_c\|_2 \leq 20$ has been taken into account. The optimization problem has been solved using sequential quadratic programming [9] (*constr* in Matlab). As starting point for the optimization problem a random matrix F, generated according to a normal distribution with zero mean and variance 0.1, was chosen. Further we select $Q = I$, $\Lambda = 0.1$, $\gamma = 100$, $\alpha = 1$. In Figs. 2–5, a resulting controller, corresponding to $\gamma = 1.47$ and $\alpha = 6.84$, is shown. The scheme has previously been investigated in [25] for identical master–slave systems. Fig. 2 shows the recovery for binary valued continuous time reference inputs or message signals, for nonidentical master-slave system. A perturbation of the element a_{11} of the A matrix $\delta a_{11} = 0.001$ is taken for the master system with respect to the nominal slave system with $A_2 = A$, $B_2 = B$, $C_2 = C$. For $\delta a_{11} = 0.01$ the original message cannot be recovered. This illustrates a difference between the synchronization scheme with reference input (1) and its autonomous case, i.e., without a message signal, considered in [27]. In the autonomous case a large parameter mismatch can be allowed such that the systems

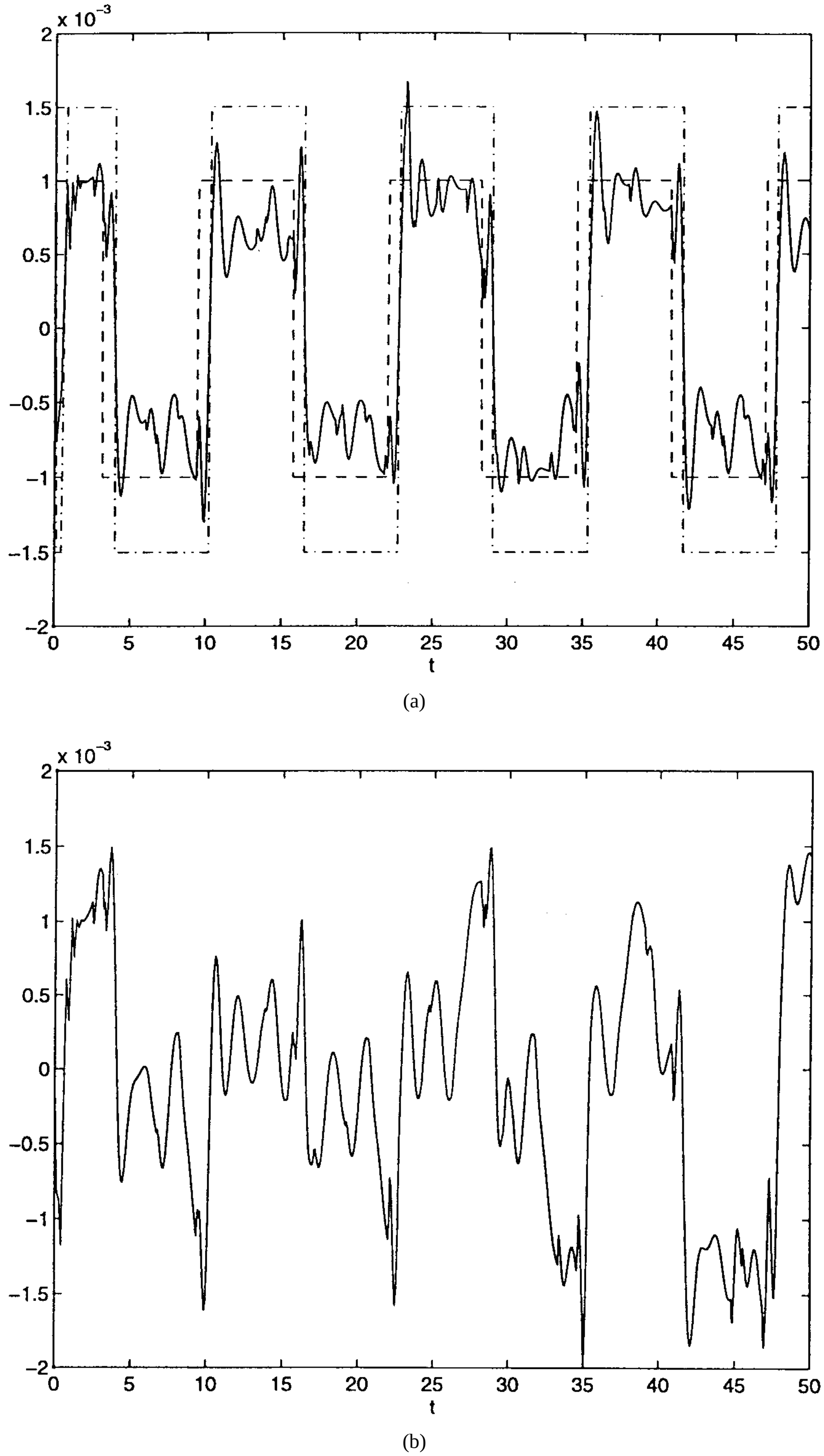
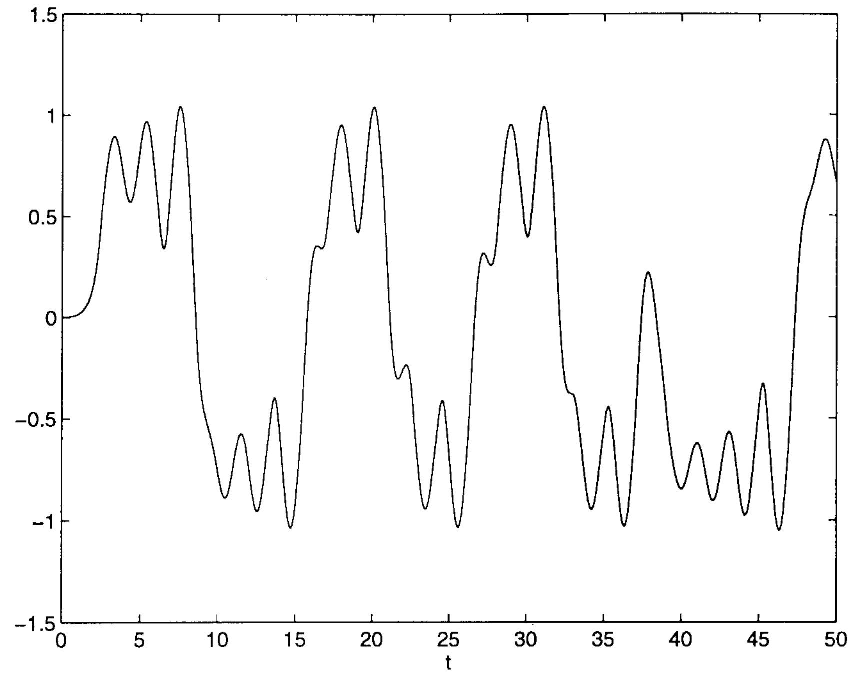
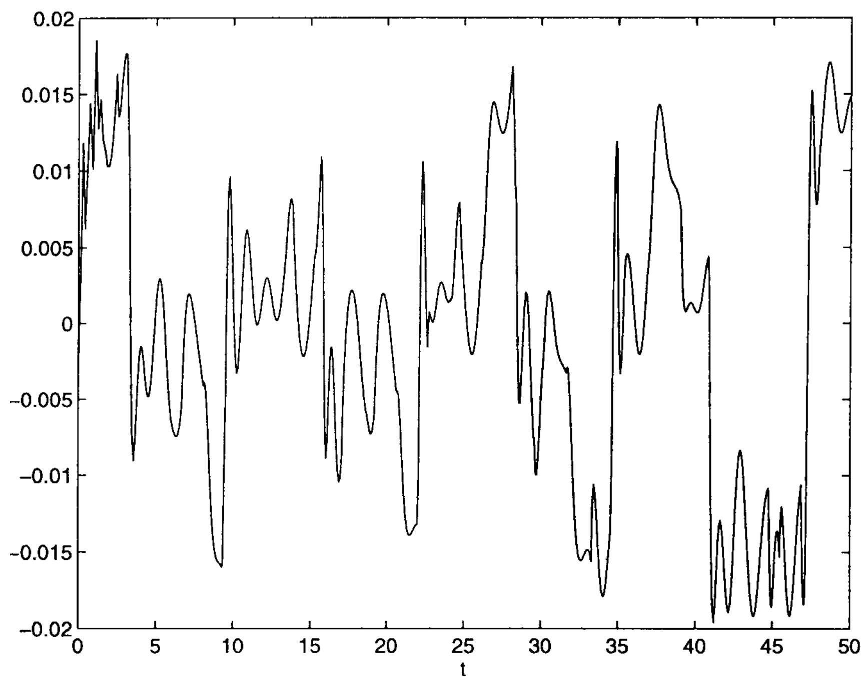


Fig. 6. Robust nonlinear H_∞ synchronization of Chua's circuit using dynamic output feedback. (a) $\beta^T e_d$ (-), scaled version of message signal sign $[\cos(0.5t)]$ (- -) and scaled version of recovered message signal sign $(\beta^T e_d)$ (-.). The parameter mismatch of the master with respect to the slave system is $\delta a_{11} = 0.001$. (b) $\beta^T e_d$ for a too large parameter mismatch $\delta a_{11} = 0.005$. The original message is not recovered in this case.

remain synchronized up to a relatively small synchronization error, even when a master system with periodic behavior or stable points is considered for a chaotic slave system. The latter is different for the nonautonomous case, as is illustrated on Fig. 4. The transmitted signals and control signals are shown on Fig. 3. Fig. 5 shows the synchronization scheme for the parameter mismatch $\delta a_{11} = 0.001$ and zero mean Gaussian channel noise with standard deviation 0.0001. The amount of noise that can be tolerated is smaller than for the case without parameter mismatch [25]. The simulations for the deterministic



(a)



(b)

Fig. 7. Dynamic output feedback (continued). (a) Transmitted signal p which is a linear combination of the state variables x_1, x_2 of the master system. Applying vector field modulation, the message signal is invisible on the chaotic carrier signal. (b) Control signal u applied to the slave system.

case were done using a Runge–Kutta integration rule [21] (*ode23* in Matlab). Stochastic systems have been simulated using an Euler integration rule [17].

The case of robust nonlinear H_∞ synchronization of Chua's circuit using dynamic output error feedback is shown on Figs. 6–8. Cryptographical issues of this scheme are discussed in [26]. The matrix H_d, together with parameters of Chua's circuit, can be chosen as a key. In the example here we take one-dimensional outputs $p, q(l = 1)$ with $H_d = [0.5; -0.5; 0]$, a one-dimensional control signal $u(m = 1)$ with $F = \beta = [1; 0; 0]$ and $D = \beta = [1; 0; 0]$. The same low pass filter $\mathcal{R}$ was chosen as for the static feedback case. The optimization problem (23) has been solved for $c = 5$, $\delta = 0.01$. An additional constraint $\|\theta_c\|_2 \leq 60$ is taken into account. A third-order SISO controller has been selected, which turned out to be the minimal order

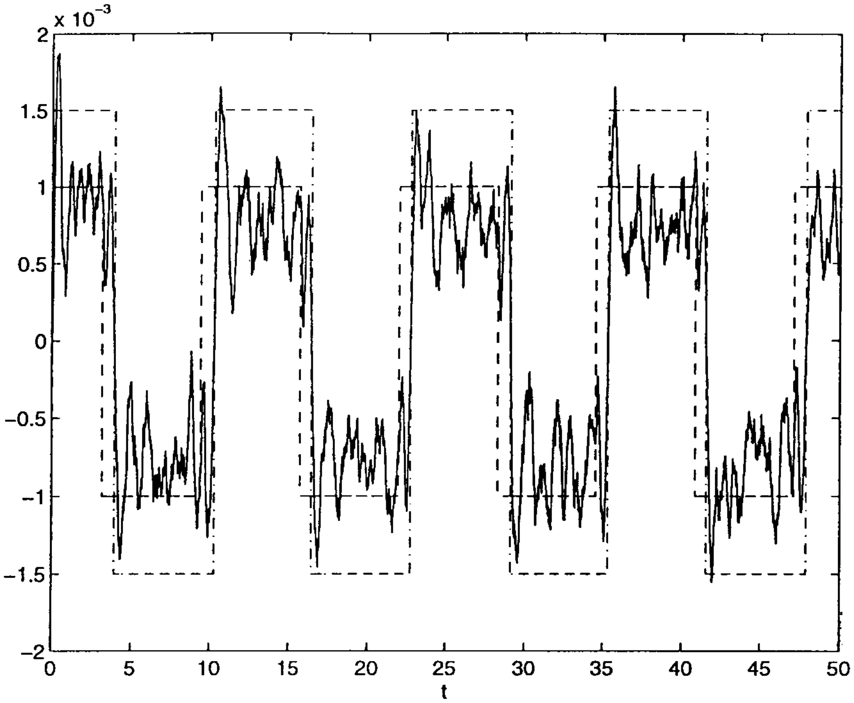


Fig. 8. Dynamic output feedback (continued). Simulation of the synchronization scheme with parameter mismatch $\delta a_{11} = 0.0001$ and zero mean white Gaussian channel noise ϵ with standard deviation 0.0001.

for achieving a good performance. As starting point for the optimization problem, random matrices were chosen for the controller according to a normal distribution with zero mean and standard deviation 0.1, $Q = I$, $\Lambda = 0.1$, $\gamma = 100$, and $\alpha = 1$. In Figs. 6–8, a resulting controller with $\gamma = 26.63$ and $\alpha = 1.59$ is shown. This scheme has been investigated for identical master–slave systems in [26]. In Fig. 6, a perturbation $\delta a_{11} = 0.001$ is considered for the master system with respect to the slave system. Perfect recovery is obtained for binary valued continuous time reference inputs, but not for a larger parameter mismatch $\delta a_{11} = 0.005$. The transmitted signal and control signal are shown on Fig. 7. Simulations with zero mean white Gaussian channel noise with standard deviation 0.0001 and parameter mismatch $\delta a_{11} = 0.0001$ are shown on Fig. 8. Hence, the performance of the full static state error feedback controller is better than for the dynamic output feedback controller, while the latter may lead to a simpler implementation of the synchronization scheme.

VII. Conclusion

The influence of parameter mismatch between master–slave Lur'e systems has been studied with respect to the method of nonlinear H_∞ synchronization. By representing the synchronization schemes in standard plant form and deriving conditions for dissipativity with finite L_2-gain, matrix inequalities have been derived. Controller design based on these matrix inequalities involves the solution of an optimization problem. The controller is rendered robust with respect to channel noise and parameter mismatch between the master–slave systems. The method further offers the possibility for incorporating channel models in the scheme. Both full static state error feedback and dynamic output error feedback have been investigated. For the latter method one can transmit a single signal, which may lead to a simpler implementation of the synchronization scheme. The full static state feedback method on the other hand has a higher performance. This has been illustrated on Chua's circuit. While in previous work we have shown that for the autonomous case a large parameter mismatch is tolerated for master–slave synchronization of the scheme up to a relatively small synchronization error, a smaller parameter mismatch is required for adequate performance of the scheme with message input.

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