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A PRIORI NONLINEAR MODEL STRUCTURE SELECTION FOR SYSTEM IDENTIFICATION

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Abstract: When performing nonlinear system identification few tools exist for the a priori nonlinear model structure selection of the nonlinear system. This paper presents a possible approach as a first step towards selecting a nonlinear system model structure, based on using the results of Lyapunov exponents, Poincaré maps and dimension techniques. The approach is illustrated by applying it to the Chua circuit, a nonlinear dynamic system exhibiting chaotic dynamic behaviour.

Keywords: Nonlinear Systems, Modelling, Control Systems Analysis, Identification

1. INTRODUCTION

The field of system identification, either in obtaining a suitable model to be used for control design or alternatively in adaptive control, is well developed for linear systems. However the linear model identified is only useful if the underlying physical process exhibits qualitatively similar dynamic behaviour to the linear model in the operating region of interest. The emphasis on linear model identification is mainly due to a lack of understanding of nonlinear systems and the fact that as yet there is no general nonlinear systems theory. There definitely is no lack of practical problems in which nonlinear systems theory would be most useful. (Gleick, 1987; Schreiber and Marek, 1982; Schreiber, 1986; Kevrekidis *et al.* 1986; IEEE, 1995). Due to the lack of familiarity with the available nonlinear systems theory in the engineering world, the widespread occurrence of nonlinear phenomena, that cannot be modelled linearly, has been widely ignored.

With the advent of the qualitative theory of dynamic systems (Thompson and Stewart, 1986) and the current interest in chaotic dynamic systems (Parker and Chua, 1989; Wigdorowitz and Petrick, 1991; Thompson and Stewart, 1986; Petrick 1989; IEEE 1993), advances have been made in developing an approach to modelling and analyzing nonlinear systems. This paper attempts to present this approach with regard to selecting a nonlinear model structure for system identification. The nonlinear analysis techniques used in this respect which have been developed in the literature are Lyapunov exponents, Poincaré maps and Dimension (Parker and Chua, 1989; Henon, 1982; Shimada and Nagashima, 1979; Farmer *et al.*, 1983; Wolf *et al.*, 1985; Benettin *et al.*, 1980; Eckmann *et al.*, 1986; and Pettis *et al.*, 1979).

The aim of this paper is to serve as a tutorial showing how these results can be used as a first step in obtaining models for system identification of practical engineering problems. This is done by applying the approach to the Chua circuit which has been extensively analyzed both analytically and experimentally (Matsumoto *et al.*, 1985; Chua *et al.*, 1986; and Petrick, 1989). The emphasis is not on the specific numerical algorithms used to implement these techniques, but rather on their application.

2. MODELLING PHILOSOPHY

The impact which deterministic models exhibiting chaotic dynamic behaviour has on modelling was discussed in (Wigdorowitz and Petrick, 1991). Here only a few essential modelling perspectives needed for selecting a model structure are outlined. It is important to have realistic expectations at the start of the modelling process. The engineer who is under the impression that modelling will give rise to the "true" system is bound to be bitterly disappointed, especially if the system exhibits a variety of different types of nonlinear dynamic behaviour in the operating region of interest. This "true" model is an esoteric entity which can never be attained in modelling physical systems (Wigdorowitz and Petrick, 1991). What can be achieved is a suitable model within the operating region of interest, useful for the application in mind. The model, whether linear or nonlinear, is always only a partial description for the system in a particular operating region under particular conditions for a specific application.

To be successful the entire modelling process should be given as much information about the system as is practical. Here a priori knowledge, experimental data and experience at modelling are crucial. The process of modelling from experimental data is known as system identification and is discussed in detail in (Ljung, 1987) and (Ljung, 1991). The framework for system identification provided in these references is used in this paper.

It is important at the outset to stress the essential differences between linear and nonlinear system identification. When referring to the dynamics of a linear system, and the identification thereof, the theory addresses the transient behaviour as the system approaches its equilibrium point, which is always a fixed point in the state space of the system. The poles or eigenvalues are used to describe this transient response of the system, and linear system identification addresses the issue of finding their values.

In nonlinear systems it is possible that the steady state of the system itself is dynamic. The trajectories settle on a structure in the state space after the transients have died away, but do not settle down to a point. Such systems cannot be represented by linear models. This paper addresses the issue of finding nonlinear model structures capable of representing such systems. The problem of modelling the transient behaviour of nonlinear systems remains, but this is beyond the scope of this paper. The reader familiar with linear systems theory is therefore warned not to attempt to draw parallels between nonlinear system parameters such as Lyapunov exponents and linear parameters like eigenvalues, as they address completely different issues.

3. THE SYSTEM IDENTIFICATION PROCESS

The system identification procedure (Ljung, 1987; Ljung, 1991) is outlined below.

1. Design an experiment to obtain the physical process input/output experimental data sets pertinent to the model application.
2. Examine the measured data. Remove trends and outliers. Apply filtering to remove measurement and process noise.
3. Construct a set of candidate models based on information from the experimental data sets. This step is the model structure identification.
4. Select a particular model from the set of candidate models in step 3 and estimate the model parameter values using the experimental data sets.
5. Evaluate how good the model is, using an objective function and the points discussed in Section 3. If the model is not satisfactory then repeat step 4 until all the candidate models have been evaluated.
6. If a satisfactory model is still not obtained in step 5 then repeat the procedure either from step 1 or step 3, depending on the problem.

This paper focuses on techniques which can aid in the model structure selection, namely step 3. At present only a possible systematic viable approach is proposed and not a general, infallible methodology. The system identification process basically amounts to repeatedly selecting a model structure, computing the best model parameter estimates and evaluating the model's properties (Section 6) to see if they are satisfactory.

4. MODEL STRUCTURE SELECTION

Before moving on to outline the contribution which Lyapunov exponents, Poincaré maps and Dimension estimates can make to the nonlinear systems identification process (Section 5), it is necessary to clarify what the model structure selection involves. The choice of a model structure must be based on an understanding of both the identification procedure, and the system to be identified. Once a model structure has been chosen, the identification procedure provides a particular model within this structure. If the identification procedure is applied correctly this model will be the best one within the structure. For a nonlinear structure, this may itself be very difficult to attain. The issue is however whether the model is good enough for the intended application. The importance of model validation can thus not be overly emphasized.

Consider the main aspects influencing the choice of a model structure:

1. What type of model is needed, nonlinear or linear, static or dynamic, distributed or lumped?

2. How large must the model set be? This question includes the issue of expected model orders and types of nonlinearities.
3. How must the model be parameterized? This involves selecting a criterion to enable measuring the closeness of the model dynamic behaviour to the physical process dynamic behaviour as model parameters are varied.

It should now be apparent that the actual identification procedure boils down to an optimization algorithm which is run on the selected model structure with the parameterization criterion as objective function. The difficulty in nonlinear system identification is selecting the model structure. Once you have a structure which can give rise to the observed dynamics, parameterization becomes a numerical problem which can be solved.

5. AN OVERVIEW OF THE TECHNIQUES CONSIDERED FOR NONLINEAR SYSTEM IDENTIFICATION

When performing system identification on nonlinear systems, it is recommended that Poincaré maps, Lyapunov exponents and dimension techniques be used in conjunction with conventional system identification and analysis techniques such as correlation analysis, frequency spectrum analysis and time history simulation. A brief summary of the pertinent aspects of each technique relating to its application in the analysis of nonlinear systems is given.

5.1 *Time history analysis*

The usefulness of time history analysis is that it can illustrate the qualitative dynamic behaviour of the system. For instance, periodicity in the waveform indicates the existence of cyclic behaviour. Should the waveform oscillate and yet appear to have no periodicity, then it may be a candidate for chaotic dynamic behaviour. However it is difficult to distinguish between quasi-periodicity and chaos solely from a time history analysis. Care must also be taken to ensure that the transient response has subsided in order to interpret steady-state nonlinear dynamic behaviour such as limit cycles, quasi-periodicity or chaos correctly.

Analysing nonlinear systems from time histories only, it is necessary to ensure that analyses are performed with all possible initial parameter values and over all possible parameter variations in the operating region of interest. It should thus be obvious that for strongly nonlinear systems time history analysis is costly and time consuming, and could be dangerous as vital areas could be missed.

5.2 *Frequency spectrum analysis*

Although a frequency spectrum may not give conclusive evidence, it can aid in defining the type of steady-state dynamic behaviour exhibited by a specific system (all parameters and inputs remaining constant). For example, oscillatory behaviour is characterized by the existence of fixed harmonics at specific frequencies, and chaotic behaviour is characterized by a bandlimited white noise frequency spectrum. The usefulness of frequency spectrum analysis is that when used in conjunction with time history analysis, it can aid in verifying the results obtained.

5.3 *Poincaré maps*

Poincaré maps are essentially a technique for the analysis of the geometric structure of attractors in phase space (Parker and Chua, 1989; Henon, 1982). This is only of interest in systems with richly nonlinear dynamics. Once it has been established that a system has to be modelled nonlinearly, Poincaré maps are useful in classifying the type of nonlinear behaviour which the system must exhibit. A precise mathematical description of Poincaré maps may be found in (Parker and Chua, 1989). Here some of the properties of these maps are described.

1. The Poincaré map is generally constructed in N-dimensional phase space by defining a $N-1$-dimensional hyperplane and studying the intersection of flow trajectories with the hyperplane. Different maps are obtained depending on the direction of intersection of the trajectories and the hyperplane.
2. The Poincaré map replaces continuous trajectories with a discrete mapping (which may be impossible to find). The plane intersection time Δt is not constant.
3. The essential idea behind the Poincaré map is to reduce the complexity of the system being studied. The mapping, which is of lower dimension than the system, retains the topological qualities of the system (it has the same qualitative properties) which simplifies analysis.

It is important to note that the Poincaré map is only valid for the analysis of structurally stable dynamics whose Poincaré map does not qualitatively change with infinitesimally small perturbations of either the system parameters or the hyperplane. For example, there is no Poincaré map for fixed-point steady-state behaviour, for if a hyperplane that intersects the fixed point were slightly perturbed, the map could no longer be reproduced. Predominantly linear behaviour is characterized by fixed-point behaviour in the steady-state for which the Poincaré map by definition is not defined. Figure 1 illustrates a Poincaré map constructed for a limit cycle. The cycle is represented by a point on the map. In mapping the limit cycle, the complexity of a three-dimensional system is reduced to a two-dimensional discrete mapping. This mapping retains all the topological properties of the system. It should also be clear from the figure that a fixed point in the state space would be unstable on the map, as mapping it would rely on the hyperplane being placed at exactly the right location. For this reason there is no Poincaré map for a fixed point.

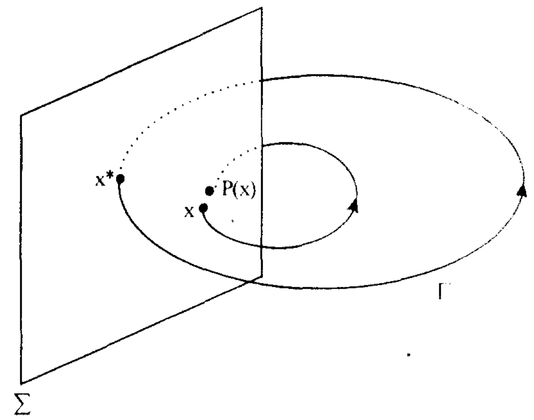

Fig.1. A Poincaré map constructed for a limit cycle (x^*) and the first mapping of a quasi-periodic cycle (x)

The usefulness of the Poincaré map is in characterizing the geometric structure of the different types of steady-state nonlinear dynamic behaviour. The system model has to produce limit cycles if the Poincaré map contains one or more simple points. Quasi-periodicity, oscillations produced by two or more modulated frequencies of oscillation, is seen on the Poincaré map as a projection of a k th order torus where k is the number of modulated frequencies in the dynamics. When the underlying dynamics are chaotic the Poincaré map exibits a fractal geometric structure as the case study in Section 7 will show. The following aspects distinguish Poincaré maps of chaotic systems from those of non-chaotic systems: the geometric object on which the trajectories lie is no longer a set of simple points or an ellipsoid, but complex. It has a very fine, delicate structure with layers within layers. This structure is characterized by a fractal dimension.

5.4 *Dimension*

Fractal dimension techniques are useful in that they can define the minimum number of first-order differential equations needed to model the dynamics of a system. This is valuable as it limits the nonlinear model structure size.

The concept of a fractal dimension requires some explanation. Dimension is intuitively understood in terms of the measurable properties with which an object can be uniquely described. A line for example has length, that is, one dimension. A plane has length and width, that is, two dimensions. In a similar way a cube will have three dimensions. A dot however cannot be measured, so assigning it a dimension of zero is plausible. In topology it is possible to construct objects which do not conform to any of the standard dimension descriptions. Take a line for example and remove the middle third. Recursively continue removing the middle third of the remaining lines. The structure which remains is a set of dots distributed over the length of the line. This structure, the cantor set, is a prime example of a geometric object which is neither a line nor a point, but something in between. These types of objects are given a fractal dimension.

Roughly speaking, a dimension provides a measure of how much space a set fills. It is a measure of the prominence of irregularities of a set when viewed at very small spatial scales (Falconer, 1990). A strange attractor, a geometric structure similar to the cantor set, underlying a chaotic dynamic system is characterized by a fractal dimension since it does not occupy the complete phase space (Thompson and Stewart, 1986). Thus systems with chaotic nonlinear dynamics are characterized by non-integer (fractal) dimension, while non-chaotic systems have integer dimension. Note that systems with dominantly linear behaviour would have dimension = 0 (fixed-point behaviour).

The current problem is to determine a common definition of dimension since at present no precise definition of a fractal exists which does not result in certain fractal sets being excluded. It is important to realize that different definitions of dimension may give different values of dimension for the same set (Falconer, 1990). The qualitative information conveyed is however the same throughout.

As the aim in this paper is to use dimension as the lower bound on the number of first-order differential equations needed to model the dynamics of a system, the choice of which definition of dimension to use depends primarily on the ease and efficiency of computation (Parker and Chua, 1989; Farmer *et al.*, 1983; Grassberger and Procaccia, 1984; and Procaccia, 1985). The definition used throughout this paper is that of Nearest Neighbour Dimension (Pettis *et al.* 1979).

5.5 *Lyapunov exponents*

Lyapunov exponents represent a reliable technique for indentifying the type of steady-state nonlinear dynamic behaviour a system exhibits (see Table 1). A reliable method exists (Shimada and Nagashima, 1979; Wolf *et al.*, 1985; and Benettin *et al.*, 1980) for computing all the Lyapunov exponents from a system model. Using the algorithm in (Wolf *et al.*, 1985) gives only the largest Lyapunov exponent which, although it limits what one can say about the system order or stability, is the largest exponent which in stable nonlinear systems indicates the type of nonlinear steady-state behaviour: chaos, periodic, or fixed-point behaviour.

Lyapunov exponents are computed by monitoring the long-term evolution ($t \rightarrow \infty$) infinitesimally small volume element comprising the initial state conditions of a model. The length of the volume element principal axis $p_i(t)$ the i th one-dimensional Lyapunov exponent according to equation (1). Here the term *principal axes* signifies the set of vectors which span the subspace in which the dynamics occur. These axes must be linearly independent, but need not be orthogonal.

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \log_2 \frac{p_i(t)}{p_0(t)} \qquad (1)$$

Table 1: Steady state identification using Lyapunov exponents

third order	fourth order	attracting set
(-,-,-)	(-,-,-,-)	fixed point
(0,-,-)	(0,-,-,-)	limit cycle
(0,0,-)	(0,0,-,-)	quasi-periodic (2)
	(0,0,0,-)	quasi-periodic (3)
(+,0,-)	(+,0,-,-)	chaos (strange attractor)
	(+,+,0,-)	hyperchaos

The units of Lyapunov exponents are information bits per second. Physically Lyapunov exponents refer to the expansion and contraction of flows in different directions in phase space. The orientation of the volume elements changes continuously with the evolution of the flow. This makes it impossible to give a well-defined direction to each exponent. Takens (1980) showed, as summarized in Table 1, that at least one Lyapunov exponent is zero if the dynamics do not exibit fixed-point steady-state dynamic behaviour. This zero corresponds to an unchanging principal axis vector tangent to the flow. In addition Takens (1980) showed that expansion in a particular direction yields positive Lyapunov exponents and contraction yields negative exponents. For a dissipative system to be bounded, that is for the dynamics to stay on some bounded geometrical structure, contraction has to outweigh expansion, giving

$$\sum_i \lambda_i < 0 \qquad (2)$$

If this sum is positive, trajectories will escape the bounded volume in phase space under consideration and hence the system is unstable and is not an attractor. The case where $\sum_i \lambda_i = 0$ is a mathematically pathological case which is practically unimportant. Non-chaotic systems do not have positive Lyapunov exponents. When a system becomes chaotic, at least one Lyapunov exponent becomes positive, which indicates an expanding flow direction. While this seems to contradict the notion of a stable system, the contraction still outweighs the expansion, implying a folding in the dynamics resulting in widely separated trajectories which merge and keep the dynamics bounded in phase space. Each positive exponent reflects a "direction" in which the system experiences repeated stretching and folding that decorrelates nearby states on the attractor. This means that the dynamic behaviour of the chaotic system with initial conditions having finite precision cannot be predicted in the long term, which is a hallmark of chaos. Thus, there exists a finite time limit beyond which quantitative prediction and hence simulation becomes meaningless. The largest positive Lyapunov exponent value is useful in quantifying this predictability margin.

More than one Lyapunov exponent is referred to as hyperchaos. As stated previously, if the system does not exhibit fixed-point steady-state dynamic behaviour, then one Lyapunov exponent must be zero. Therefore a stable system must be at least third-order if it is to exhibit chaotic dynamic behaviour, since it has to have three Lyapunov exponents, namely one negative, one positive and one zero. Table 1 shows the possible combinations of Lyapunov exponents in third- and fourth-order autonomous systems, and the corresponding type of steady-state dynamic behaviour.

6. PRACTICAL SYSTEM IDENTIFICATION USING POINCARÉ MAPS, LYAPUNOV EXPONENTS AND DIMENSION

Having introduced Lyapunov exponents and Poincaré maps, as well as dimension techniques, it is now shown how these techniques can be used for nonlinear system identification. The physical process used to illustrate the approach is the Chua circuit (Matsumoto *et al.*, 1985; and Chua *et al.*, 1986). This is a time-invariant autonomous system. This system is an analogue electronic system, simple enough to present all the required nonlinear phenomena needed to illustrate the process of identifying general nonlinear system, without clouding the issues with the need to understand the physical properties needed in modelling a complex system.

The first point to emphasize is that for nonlinear system identification, the first step is to model the physical process in the neighbourhood of an operating point (equilibrium point of the system) or in a bounded region of the state space. Models must then be obtained within each region of the state space within the operating region of interest. This is particularly important when a change in the type of qualitative dynamic behaviour occurs for different model parameter values. To begin, the Chua circuit output time history response as shown in Fig. 2 is considered for an initial perturbation in the state variable values. Initially the dynamic behaviour of the system must be analyzed to determine whether it exhibits nonlinear dynamic behaviour which can be qualitatively reproduced by a linear model. If this is the case then a conventional linear system identification should be used. As a first approximation a system can be modelled linearly in the neighbourhood of an operating point if its steady-state dynamic behaviour is of fixed-point type at the operating point. If the dynamics exibit steady-state cycling then this cannot be reproduced by a linear model and a nonlinear model will have to be resorted to. The steady state Chua circuit output response (Fig. 2) is definitely cyclic.

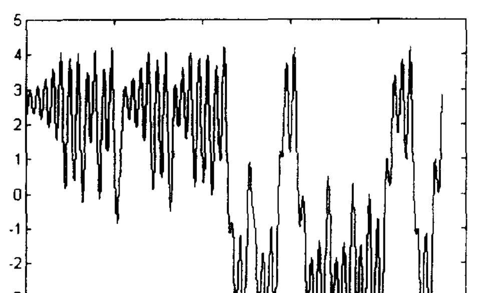

Fig. 2. Experimental Chua Circuit Output Response Waveform

Thus the system must be modelled by a nonlinear model. The next question is what type of nonlinear behaviour does the system exhibit? The time history in Fig. 2 appears regular, but is there any periodicity in this oscillatory behaviour? This question has to be answered in order to establish whether this is periodic, quasi-periodic or chaotic behaviour. From a frequency analysis of the waveform in Fig. 2, the power spectral density shown in Fig. 3 indicates aperiodic behaviour. The power spectrum appears noiselike over a finite bandwidth which in conjunction with the time history response indicates that this potentially is a chaotic dynamic system. The results though are inconclusive.

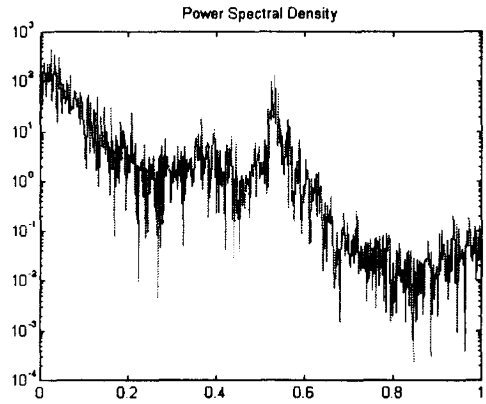

Fig. 3. Power Spectrum of the Output Waveform in Fig. 2

The nature of the steady-state dynamics has been provisionally identified, but will have to be confirmed by further analysis. The basis for all further analysis is the idea of attractor reconstruction, the philosophy and mathematical basis of which is given in (Parker and Chua, 1989; Takens, 1980). This technique topologically reconstructs the system attractor (the phase space structure that the trajectories approach in steady state) from experimental time series data of a single state variable. The first step in reconstructing an attractor is to determine the dimension of the attractor. The process of establishing this parameter ensures that the reconstructed attractor is an embedding of the original dynamic system. This means that the reconstructed system has retained the topological properties of the original system (loosely speaking is qualitatively identical), that is the original and reconstructed attractor both have the same Lyapunov exponents and dimension.

6.1 *Dimension estimation from the time series data*

In the process of computing the fractal dimension of a chaotic attractor from the time series data of one state variable, the first step is to determine an embedding of the attractor. An embedding is obtained by choosing equally spaced time samples of the time series of only one state variable to represent the reconstructed state variables of the system. For example, suppose you have the time series data set $\{x(t_1), x(t_2), \ldots x(t_N)\}$ of a single state variable on an attractor, an embedded attractor may be obtained by constructing delay coordinates $\{x(t_i), x(t_{i+\tau}), \ldots x(t_{i+m\tau})\}$ where τ is called the sample delay time and m the embedding dimension. The parameter τ must be chosen so as to ensure that the points used to represent the different phase space coordinates are not too close in value in order to avoid any periodicity in the original waveform and not too far apart such that the coordinates are uncorrelated on the chaotic attractor, if the attractor is chaotic. Experience, has found the technique to be insensitive to the choice of τ.

The embedding dimension is found in the process of computing the dimension of the attractor as shown in Fig. 4. Initially the system embedding dimension is assumed to be one. The system dimension is then calculated using any of the dimension definitions such as $D_{\text{nearest neighbour}}$ or $D_{\text{correlation}}$ (Parker and Chua, 1989; Farmer *et al.*, 1983; Grassberger and Procaccia, 1984; Procaccia, 1985; and Pettis *et al.*, 1979).

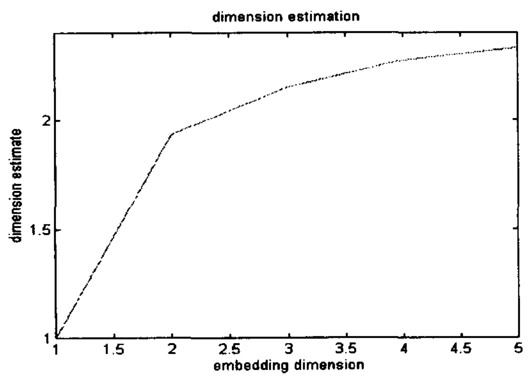

Fig.4. An estimation of the dimension of the Chua attractor from time series data. The nearest neighbour dimension (Pettis *et al.*, 1979) is estimated as the embedding dimension increases. As soon as the estimate settles to a constant value an embedding has been obtained. The dimension estimate is taken to be the fractal dimension.

In Fig.4. the dimension definition implemented was $D_{\text{nearest neighbour}}$ (Pettis *et al.*, 1979). The embedding dimension is now increased and the dimension is again computed. This process continues until the dimension estimate slope settles to a value which is no longer sufficient to reach the next full integer. The value at which the dimension estimate settles down is the dimension of the attractor and the corresponding embedding dimension is the embedding of the attractor. From Fig. 4., the Chua system at the particular operating point has a fractal dimension of between *2* and *3*, which confirms that the steady-state dynamics are indeed chaotic. It further shows that the type of steady-state behaviour observed cannot be modelled using less than three first-order nonlinear differential equations. Hence an indication of the type of dynamic behaviour and the minimum number of first-order nonlinear differential equations required to reproduce this type of dynamic behaviour has been obtained.

6.2 *Estimating the largest Lyapunov exponent from time series data*

In order to verify the conclusions (Section 6.1) as to the type of dynamic behaviour and to determine the rate of decorrelation of nearby trajectories for the chaotic dynamics expected, the largest Lyapunov exponent using any one of the algorithms in (Wolf *et al.*, 1985; Eckmann *et al.*, 1986) is computed. These algorithms are based on the idea of time delay attractor reconstruction introduced in Section 6.1. Having first computed the dimension of the attractor (Section 6.1), the embedding dimension and sample delay time are already known for the estimation of the largest Lyapunov exponent.

Recall how the largest Lyapunov exponent may be interpreted: a positive Lyapunov exponent implies chaos. If the largest Lyapunov exponent is zero, this points to periodic or quasi-periodic behaviour, while for fixed-point behaviour the largest exponent is negative. In addition the largest Lyapunov exponent defines the rate of information increase (positive value) or decrease (negative value) in a system.

The value computed for the experimental Chua circuit waveform (Fig. 2) was *0.7*. The units of the Lyapunov exponent are bits per second. This implies that measuring the output waveform to an accuracy of *16* bits, the uncertainty in the measurement implies that from this measurement the uncertainty in future predictions with a perfect model for the trajectory would be the size of the entire attractor after $16/0.7=22.86$ seconds.

6.3 *Analyzing the geometric structure of the attractor*

Having now determined the type of steady-state dynamic behaviour, the minimal order of the model, as well as the limits of predictability of a model if the system is chaotic, more insight into the geometric structure of the attractor in phase space on which the dynamics occur is needed to get closer to identifying the model structure. This may be provided by phase plane portraits and Poincaré maps. To begin with, the phase portraits of the embedded attractor are investigated. These can be drawn while the largest Lyapunov exponent is computed as suggested in (Wolf *et al.*, 1985). The resulting phase portrait is shown in Fig. 5.

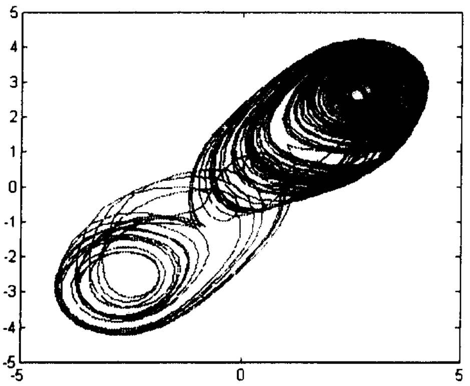

Fig. 5. Phase portrait of the reconstructed attractor

It is known from the dimension estimate, that for the Poincaré maps a 3 dimensional attractor should be considered, which was done. All the Poincaré crossections which were used to inspect the attractor structure cannot be presented here. The interested reader is referred to (Petrick and Wigdorowitz, 1993) or (Matsumoto *et al.* 1985) for samples of Poincaré maps of this system.

7. MODELLING

To summarize the model structure identification at this stage, the dimension has provided a minimal model order, Lyapunov exponents have given a measure of the predictability limitations of a model and conclusively determined the type of steady state dynamic behaviour and Poincaré maps and phase portraits have given insight into the attractor phase portrait.

This information provides the key to modelling. The essence of the nonlinear model structure identification amounts to determining a nonlinearity with which the minimal model order can give the type of state space geometrical structure seen on the Poincaré maps and phase portrait (Fig. 5). The emphasis and key to nonlinear model structure identification is to use a qualitative approach. For chaotic dynamic systems this is a necessary condition since quantitative agreement is impossible in the long term (Wigdorowitz and Petrick, 1991; Thompson and Stewart, 1986; and Petrick, 1989) (22.86 seconds for the Chua circuit!).

Consider the Chua double scroll attractor in Fig. 5. What state-space structure underlies this attractor? Can this type of structure be generated by only one equilibrium point? It is known that a stable equilibrium point itself can either yield fixed-point behaviour, limit cycling, quasi-periodicity and perhaps chaos, but this will not be on a structure as complex as the double scroll shown in Fig. 5. Such a phase portrait would be expected to be constituted of several interacting equilibrium points. The question is how many and where are they? The phase portrait in Fig. 5 shows oscillations occuring mainly around two points in the state space at approximately $\pm(2.7,0,-2.7)$. This implies that there must be two equilibrium points at these locations. These equilibrium points however do not have fixed-point type behaviour. They probably have undergone limit cycling. The next question is how do these equilibrium points interact, or in other words what makes trajectories jump from the basin of attraction of one of these points to that of the other? Closer investigation of the Poincaré maps and phase portrait around the point $(0,0,0)$ indicates that while trajectories may come close to this point, they always depart from it either to the region around $(2.7,0,-2.7)$ or $(-2.7,0,2.7)$. This seems to indicate that there is probably another unstable equilibrium point around $(0,0,0)$.

There thus are three equilibrium points, two stable, but having undergone a Hopf bifurcation, and one unstable. The two stable equilibrium points are approximately at $(2.7,0,-2.7)$ and $(-2.7,0,2.7)$ and the unstable equilibrium point at approximately $(0,0,0)$. Now a nonlinearity needs to be found that can produce this scenario in the phase space. Denote the three state variables required as x,y and z. The following structure is proposed as a first attempt:

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z + bh(x)\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + a_{23}z\\ \frac{dz}{dt} &= a_{31}x + a_{32}y + a_{33}z\end{aligned} \tag{3}$$

where $h(x)$ is a nonlinearity. The idea is to select a model structure with the minimum number of nonlinearities for simplicity and ease of computation in estimating the model parameters. From the required equilibrium points it follows that $a_{31} = a_{33} = 0$ and $a_{21} = a_{23}$. This gives a simplified model structure

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + a_{13}z + bh(x)\\ \frac{dy}{dt} &= a_{21}(x+z) + a_{22}y\\ \frac{dz}{dt} &= a_{32}y\end{aligned} \tag{4}$$

It is also known that the nonlinearity has to be equal to the line $h(x) = -x$ at $x = y = z = 0$ and at of ways in which this can be achieved, as shown in Fig. 6.

Having thus determined a possible model structure the next step is to determine the slopes of the nonlinearity $h(x)$, and the parameters $a_{11}, a_{12}, \ldots, a_{33}$ and b such that the system equations qualitatively exhibit the required steady-state dynamic behaviour. For a non-chaotic system a quantitative comparison between the model and physical process should also be done. This stage in modelling the system is complex. In (Petrick, 1989) this work was done using a piecewise linear nonlinearity $h(x)$ implementation and found very sensitive to exact parameter values.

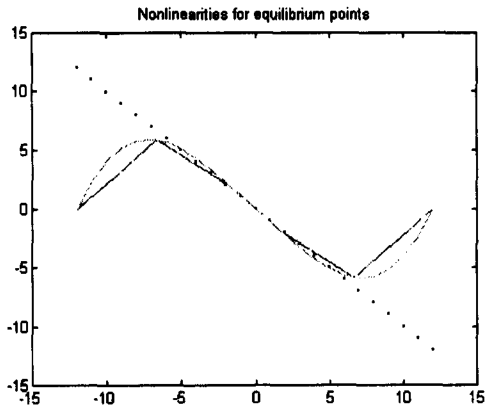

Fig. 6. Various nonlinearities able to give the required equilibrium points

Up to now all the modelling done has focused on finding a structure able to give a geometrical structure in phase space. The time characteristics of the nonlinear oscillation have thus far been ignored. The Lyapunov exponent value however is a measure of the predictability margin representing the time characteristics. Having succeeded in finding a structure that gives the required qualitative dynamics, the model time is now scaled such that the magnitudes of the model Lyapunov exponents and experimentally determined Lyapunov exponents agree.

While this exercise may appear contrived since a working model of the Chua circuit already exists, it has been illustrated how one could possibly approach the nonlinear model structure identification using techniques from nonlinear system analysis. This approach has been successfully used other nonlinear systems such as the van der Pol oscillator.

8. CONCLUSION

It has been illustrated how using Poincaré maps, Lyapunov exponents and dimension techniques, it is possible to determine nonlinear model structures for systems that cannot be represented by linear models. Furthermore these techniques provide a sound basis for determining whether to use linear or nonlinear model structures. Although the example is very simple, it is believed that the approach presented in this paper provides a reliable and time effective method for nonlinear model structure identification for systems whose dynamics lie on low-order dimensional attractors.

9. REFERENCES

Benettin, G. , Galgani, L., Giorgilli, A. and Strelcyn, J.M. (1980). Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems: a method for computing all of them. Part 2: Numerical Application. *Meccanica,* pp. 21-30 March 1980.

Chua, L.O., Komuro, M. and Matsumoto, T. (1986). The double scroll family - parts 1 and 2. *IEEE Transactions on Circuits and Systems*, **CAS-33, No. 11**, pp. 1072-1118.

Eckmann, J.P., Kamphorst, S.O., Ruelle, D. and Ciliberto, S. (1986). Lyapunov exponents from time series. *Phys. Rev. A*, **34**, pp. 4971-4979.

Falconer, K. (1990). *Fractal Geometry, Mathematical Foundations and Applications.* John Wiley & Sons.

Farmer, J.D., Ott, E. and Yorke, J.A. (1983) The dimension of chaotic attractors. *Physica D*, **Vol. 7**, pp. 153-180.

Gleick, J. (1987). *Chaos: Making a New Science.* Penguin.

Grassberger, P. and Procaccia, I. (1984). Dimensions and entropies of strange attractors from a fluctuating dynamics approach. *Physica D*, **Vol.13**, pp. 34-54.

Henon, M. (1982) On the numerical computation of Poincaré maps, *Physica D*, **Vol. 5**, pp. 412-414.

IEEE (1993). Special issue on Chaos in Nonlinear Electronic Circuits. *IEEE Transactions on Circuits and Systems.* **Vol. 40, No. 10**, pp. 640-886.

IEEE (1995). Special issue on Nonlinear Phenomena.in Power Systems. *IEEE Proceedings.* **Vol. 85, No. 11**, pp. 1439-1573.

Kevrekidis, I.G., Aris, R. and Schmidt, L.D. (1986) The stirred tank forced. *Chemical Engineering Science* **Vol. 41**, p. 1549.

Ljung, L. (1987) *System Identification: Theory for the user*, Prentice Hall.

Ljung, L. (1991) *Matlab System Identification Toolbox User's Guide*, Mathworks.

Matsumoto, T., Chua, L.O. and Komuro, M. (1985) The double scroll. *IEEE Transactions on Circuits and Systems*, **Vol. Cas-32, No. 8**, pp. 798-817.

Parker, T.S. and Chua, L.O. (1989) *Practical Numerical Algorithms for Chaotic Systems* Springer-Verlag, New York .

Pettis, K.W., Bailey, T.A., Jain, A.K. and Dubes, R.C. (1979) An intrinsic dimension estimator from near-neighbour information. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. **Vol. PAMI-1,** No. 1, pp. 25-37

Petrick, M.H. (1989) Chaos: a newly discovered nonlinear phenomenon, *Final year project report 47P/89* University of the Witwatersrand.

Petrick, M.H., and Wigdorowitz, B. (1993) The application of Poincaré maps, Lyapunov exponents and dimension techniques to engineering problems*Trans Inst MC* **Vol. 15** No. 4, pp. 178-187

Procaccia, I. (1985) The static and dynamic invariants that characterize chaos and the relations between them in theory and experiments. *Physica Scripta*, **Vol. T9**, pp. 40-46

Schreiber, I. and Marek, M. (1982) Strange reactors in Ccoupled reaction - diffusion cells *Physica 5D* p. 258

Schreiber, I. (1986) Periodic and aperiodic regimes in coupled dissipative chemical oscillators *Journal of Statistical Physics* **Vol. 43**, p. 489

Shimada, I. and Nagashima, T. (1979) A numerical approach to the ergodic problem of dissipative dynamical systems, *Progress in Theoretical Physics* **Vol 61** No 6, pp. 1605-1616

Takens, F. (1980) Detecting strange attractors in turbulence. *Lecture Notes in Mathematics* Springer-Verlag, New York, pp. 366-381.

Thompson, J.M.T. and Stewart, H.B. (1986) *Nonlinear Dynamics and Chaos: Geometrical Methods for Engineers and Scientists*. John Wiley and Sons.

Wigdorowitz, B. and Petrick, M.H. (1991).Modelling concepts arising from an investigation into a chaotic system. *Mathematical and Computer Modelling*, **Vol. 15,** No. 8, pp. 1-16.

Wolf, A., Swift, J.B., Swinney, H.L. and Vastano, J.A. (1985) Determining Lyapunov exponents from a time series. *Physica D*, **Vol 16,** pp. 285-317.