Experimental verification of horseshoes from electronic circuits

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Poincaré maps are an important tool in analysing the behaviour of nonlinear dynamical systems. If the system to be investigated is an electronic circuit or can be modelled by an electronic circuit, these maps can be visualized on an oscilloscope thereby facilitating real-time investigations. In this paper, sequences of return maps eventually leading to horseshoes are described. These maps are experimentally taken both from non-autonomous and autonomous circuits.

1. Introduction

Electronic circuits offer an excellent possibility, to study complex dynamics in real-time in an experimental set-up. This is due to the availability of almost ideal components – at least at the low-to-medium frequency range – enabling the construction of precise models for a set of nonlinear ODEs. Therefore, such electronic circuits may either be a specialized nonlinear analogue computer simulating a system governed by set of ODEs at much higher speed than any digital computer would be able to or they may be the object of studies of their dynamic behaviour on their own right. An excellent example for the latter case is Chua's circuit (this volume) and, for example, the cellular neural network (CNN) (Chua & Yang 1988). The detailed theoretical background and evolution of both topics will not be dealt with here, but rather the measurements taken on these circuits, especially the visualization of horseshoes one can find in these systems, when being operated with the appropriate parameters, will be described. Three different circuits together with the corresponding measurement set-up will be described:

(i) a two-cell CNN with sinusoidal excitation;
(ii) a three-cell autonomous CNN;
(iii) Chua's circuit (autonomous).

All three circuits are hooked up with resistors, capacitors, diodes and operational amplifiers only, thereby avoiding inductances, which are deviating from the ideal behaviour more than the other elements, at least at low frequencies. This reduces the problem of sensing the state variables to the measurement of capacitor voltages only.

2. Non-autonomous circuits

Cellular neural networks (CNNs) (Chua & Yang 1988) are continuous-time nonlinear dynamical systems, which easily can be implemented in form of a microelectronic

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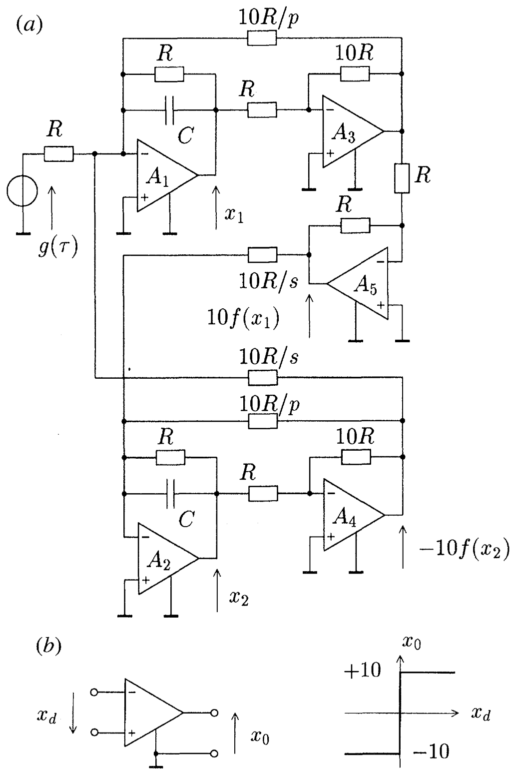

Figure 1. Two-cell anti-reciprocal CNN implementation.

circuit. To demonstrate horseshoes experimentally a two-cell anti-reciprocal CNN, as described by the following state equations:

$$\frac{\mathrm{d}x_1(\tau)}{\mathrm{d}\tau} = -x_1(\tau) + pf(x_1(\tau)) - sf(x_2(\tau)) + g(\tau), \tag{1}$$

$$\frac{\mathrm{d}x_2(\tau)}{\mathrm{d}\tau} = -x_2(\tau) + sf(x_1(\tau)) + pf(x_2(\tau)) \tag{2}$$

with $p, s \in R^+$ and

$$f(x_i(\tau)) = \tfrac{1}{2}(|x_i(\tau)+1| - |x_i(\tau)-1|), \qquad i = 1, 2, \tag{3}$$

$$g(\tau) = A \sin 2\pi\tau/T \tag{4}$$

has been implemented. The circuit given in figure 1*a* has been derived using a proper denormalization procedure (table II in Nossek *et al.* 1992), where a state reduction factor of 10 has been used to make sure, that the operational amplifiers A_1 and A_2 and A_5 stay always in the linear part of their transfer characteristic (figure 1*b*), while the saturation parts of (3) are provided by saturation of the outputs of A_3 and A_4. Due to this assumption, the capacitor state voltage x_1 and x_2 are also present at the output of A_1 and A_2 respectively. Analysis of the circuit in figure 1*a* with values of $R = 1$ and $C = 1$ will exactly result in (1)–(3). Choosing $p = 2.0$, $s = 1.2$, $A = 4.04$ and $T = 4$ (Zou & Nossek 1991) the circuit becomes chaotic and the Lady's shoe attractor can be observed (Zou & Nossek 1992). In addition to visualize the chaotic attractor of the second order system in two-dimensional state space on

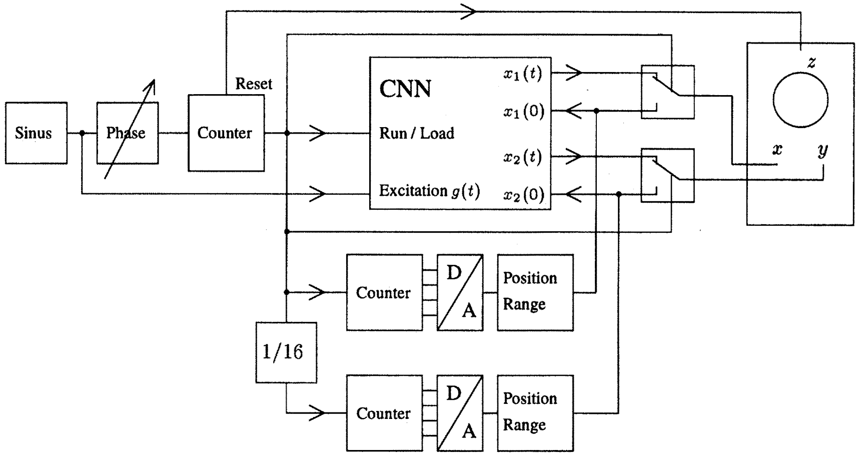

Figure 2. Experimental set-up for visualizing Poincaré return maps in non-autonomous circuits.

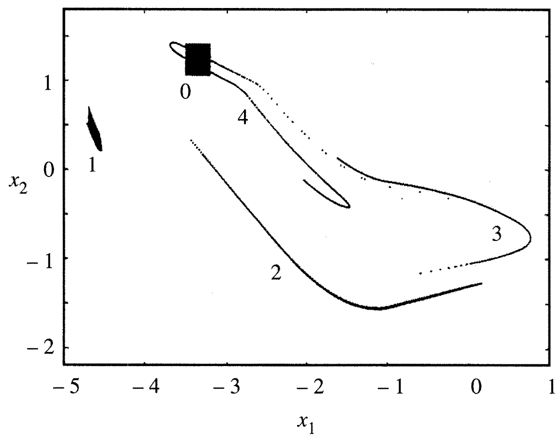

Figure 3. Simulation of Poincaré return maps.

the display of an oscilloscope, it is possible visualizing horseshoes by applying the following procedure:

(i) Provide a set of initial conditions for both state variables $x_1(0)$ and $x_2(0)$ covering a rectangle in state space. In the experimental set-up in figure 2 there are 16×16 distinct initial conditions, provided by the two 4-bit digital-to-analogue converters. This rectangle can be scaled and shifted wiht the aid of simple opamp circuitry within the modules of 'position/range adjustment' for x- and y-coordinate on the oscilloscope.

(ii) After having chosen the position and size of the rectangle of initial conditions let the system under test evolve over time beginning with the first initial state. Take snapshots of the system state vector synchronized with the excitation signal by modulating the brightness of the oscilloscope beam (z-axis input) accordingly. Then take the next initial state (controlled by the two 4-bit counters) and take again a series of snapshots (Poincaré return maps) as before. Proceed till the 256th initial state has been applied.

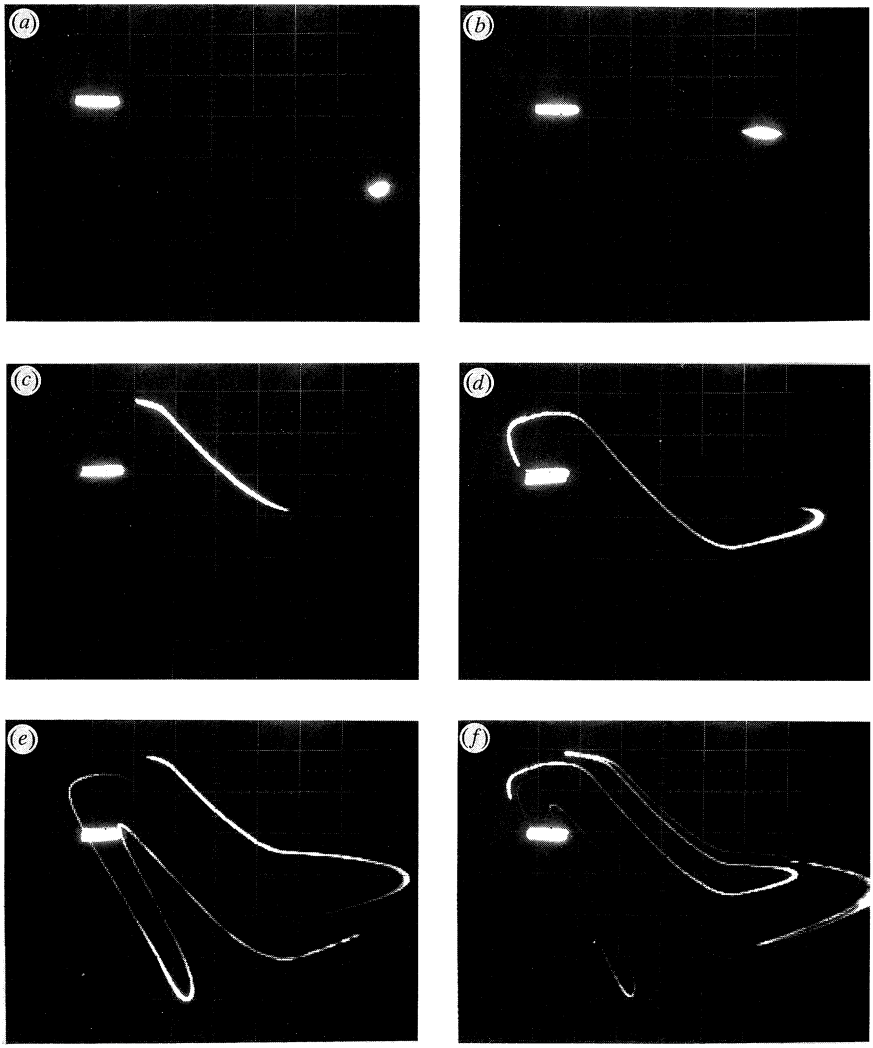

Figure 4. Initial rectangle with first to sixth return map finally showing a horseshoe (*a*)–(*f*).

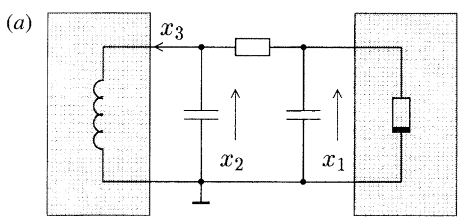

Figure 5. (*a*) Chua's circuit.

(iii) Repeat the whole procedure periodically. This will give a quasi-stationary image on the display. By slowly moving around the rectangle and scaling it properly, one could find horseshoes in real-time. With the adjustable counter one could pick the specific return map (snapshot), which actually shows the horseshoe with the initial rectangle.

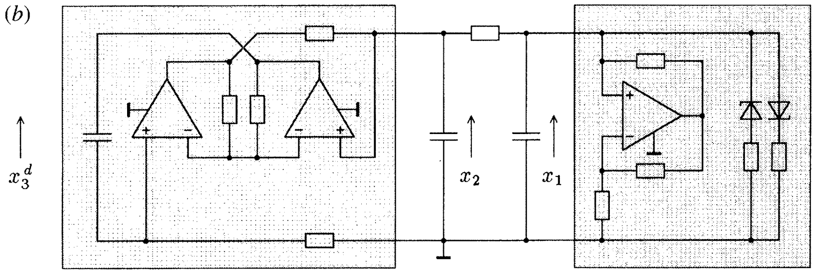

Figure 5. (*b*) Microelectronic implementation of the model.

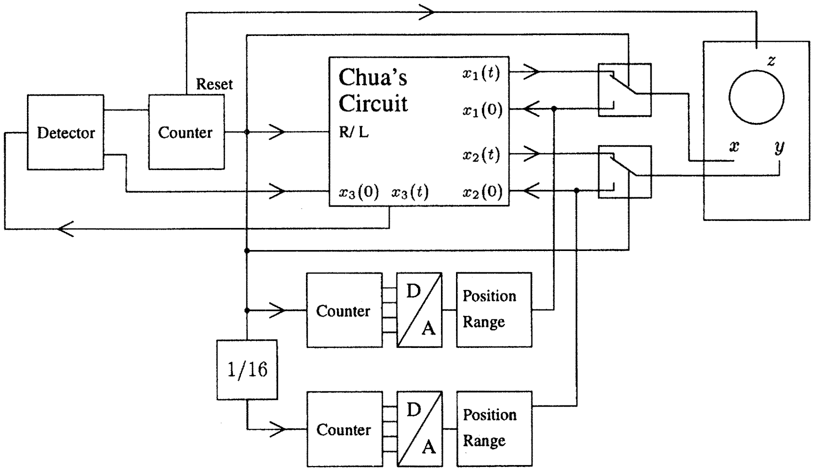

Figure 6. Experimental set-up for visualizing Poincaré return maps in autonomous circuits.

Figure 3 shows a simulation of the procedure, where the snapshot number 0 represents the initial condition and the snapshots up to number 4 are displayed, showing the first occurrence of a horseshoe. Figure 4 shows a series of photographs taken from the display of an oscilloscope actually running the experiment shown in figure 2. There always the initial conditions together with one snapshot (from the first to the fourth one) is shown. In this way experimental verification and real-time investigation of complex dynamic behaviour is possible.

3. Autonomous circuits

Two examples are considered here. One is a three-cell CNN (Zou & Nossek 1993) governed by the following equations:

$$\dot{x}_1 = -x_1 + p_1 f(x_1) - s f(x_2) - s f(x_3), \tag{5}$$

$$\dot{x}_2 = -x_2 - s f(x_1) + p_2 f(x_2) - r f(x_3), \tag{6}$$

$$\dot{x}_3 = -x_3 - s f(x_1) + r f(x_2) + p_3 f(x_3), \tag{7}$$

where $f(\cdot)$ is defined by (3) and $p_1, p_2 > 1$, $p_3 \geqslant 1$, $r, s > 0$.

The circuit implementation follows the same lines as demonstrated with the circuit in figure 2 for (5)–(7), and shows a double scroll-like attractor.

The second example is a RC-active version of Chua's circuit (figure 5). the equations of which are given in Chua's contribution to this volume.

The visualization of horseshoes for these autonomous circuits is done in the same way as in the non-autonomous case, with the exception that the precise instant, at which a snapshot has to be taken, is now not determined by the period of the excitation but by the state crossing a certain plane in state space (carried out by the 'detector'). In the set-up depicted in figure 6 this plane is parallel to the x_1 and x_2 coordinates.

4. Conclusions

Electronic circuits can serve as excellent models for simulating nonlinear dynamical systems. Thereby, they can enable experimental investigations, to be carried out in real-time. This way the first experimental observation of horseshoes has been achieved.

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