Chaotic Dynamics of the Driven Chua's Circuit

K. Murali and M. Lakshmanan

Abstract— **The nonautonomous version of Chua's circuit exhibits a wide variety of bifurcation phenomena and chaotic structures depending upon the strength of the forcing parameters. In addition, control of chaos can be effected in this circuit. We also point out that synchronization of chaos is possible in this system, which can be effectively used for secure communication.**

I. Introduction

THE STUDY OF nonlinear circuits is a convenient framework to undertake systematic exploration of fundamental mechanisms underlying the onset of chaos. In this connection, in a recent series of works we have studied the dynamics of a very simple nonlinear electronic circuit, namely the driven Chua's circuit [1]–[5]. Since its discovery in 1984 [6], [7], Chua's autonomous circuit has been studied extensively. It is an extremely simple system and yet it exhibits the complex dynamics of bifurcation and chaos. In addition, under the influence of an external periodic signal Chua's autonomous circuit turns out to be a veritable black box to study various bifurcation sequences, chaotic structures, controlling of chaos, synchronization of chaos, and so on. In this paper, we briefly report the various phenomena associated with this driven Chua's circuit.

II. Chaos in the Driven Chua's Circuit

The experimental circuit realization of the driven Chua's circuit is shown in Fig. 1. In this circuit L_1, L_2, C_1, C_2, and R are all linear elements. A single nonlinear element N is employed in the circuit; it is a piecewise-linear resistor (subcircuit N) called Chua's diode [13]. By applying Kirchoff's laws to the circuit of Fig. 1, the following circuit equations are derived:

$$
\begin{aligned}
C_1\frac{dV_{C1}}{dt} &= (1/R)(V_{C2}-V_{C1}) - g(V_{C1})\\
C_2\frac{dV_{C2}}{dt} &= (1/R)(V_{C1}-V_{C2}) + i_{L2} - i_{L1}\\
L_1\frac{di_{L1}}{dt} &= V_{C2} - (F_1\sin\omega_1 t + F_2\sin\omega_2 t)\\
L_2\frac{di_{L2}}{dt} &= -V_{C2}. \qquad (1)
\end{aligned}
$$

Here V_{C1}, V_{C2}, i_{L1}, and i_{L2} are the voltage across C_1, the voltage across C_2, current through L_1, and current through L_2, respectively. Here $F_1\sin\omega_1 t$ and $F_2\sin\omega_2 t$ are the two sinusoidal signals. The term $g(V_{C1})$ provides necessary nonlinearity for the circuit's chaotic behavior [1]–[7]. When the second inductor L_2 is absent and the external forces F_1 and F_2 are zero ($F_1 = F_2 = 0$), we have the standard Chua's autonomous circuit exhibiting a chaotic double-scroll Chua's attractor. We will consider the nonlinear dynamics of this nonautonomous circuit.

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A. Effect of Sinusoidal Excitation on the Fixed-Point Attractor [1], [2]

To start with we consider the effect of the single sinusoidal force ($F_1 \neq 0,\ F_2 = 0$). We fix the circuit parameters L_i, L_2, C_1, C_2, and R at the values 80 mH, 13 mH, 0.017 μF, 1.25 μF, and 1310 Ω, respectively, and study the system behavior in the $F_1 - f_1$ parameter plane. Particularly the amplitude F_1 is varied from 0 to 800 mV and the frequency $f_1(=\omega_1/2\pi)$ from 800 to 1500 Hz. In the autonomous case ($F_1 = 0$), for this choice of circuit parameters, the system exhibits the usual fixed-point attractor.

Based on the extensive experimental measurements of voltage changes across the capacitors C_1 and C_2, for different forcing parameter values (F_1 and f_1 with $F_2 = 0$), a profile of bifurcation diagram in the $F_1 - f_1$ plane has been drawn as in Fig. 2. This figure depicts the kind of behavior that this circuit admits, for each value of the driving amplitude F_1 and frequency f_1. In Fig. 2, the numbers indicate the period of the observed attractors and the shaded regions (Ch_1, Ch_2, and Ch_3) represent chaos. From this figure one can easily identify the regions of period-doubling bifurcation, windows, period-adding sequences, and boundary region [1], [2]. A typical chaotic attractor projected onto the $V_{C1} - V_{C2}$ plane and its corresponding Poincaré map for the region Ch_2 are depicted in Fig. 3. In addition, this circuit admits period-adding bifurcations, intermittency route to chaos, hysteresis and coexistence of multiple attractors [1], [2].

B. Double-Scroll Chua's Attractor and Devil's Staircase Structures [3]

In a second set of experiments, we carried out the investigation (still with $F_2 = 0$) by fixing the circuit parameters, as $C_1 = 600$ pF, $C_2 = 0.005$ μF, $L_1 = 2.8$ mH, $L_2 = 10$ mH, $R = 1430$ Ω, and varying the amplitude F_1 from 0 to 0.2 V and the frequency $f_1(=\omega_1/2\pi)$ in the range 24 to 150 Hz. In the autonomous case ($F_1 = 0$), we note that this circuit admits the familiar double-scroll Chua's attractor. By varying the amplitude F_1 (still with $F_2 = 0$) and frequency f_1, reverse bifurcations, quasi-periodic route to chaos, phase lockings,

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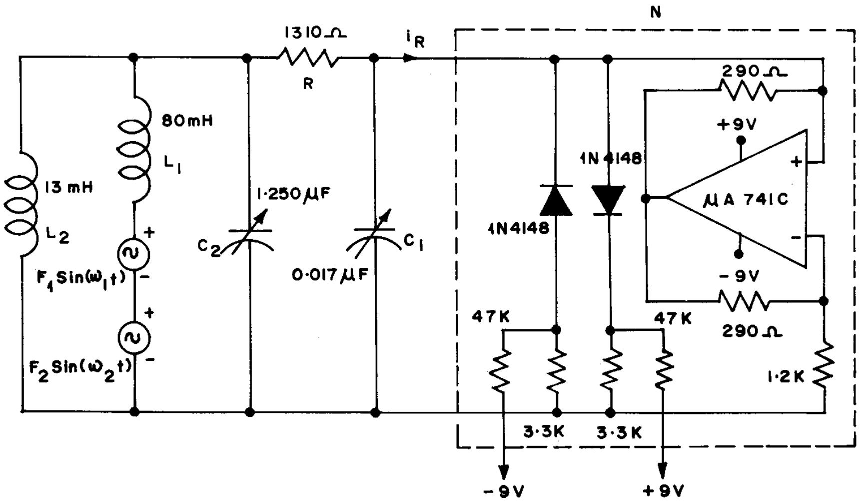


Fig. 1. Circuit realization of the driven Chua's circuit.

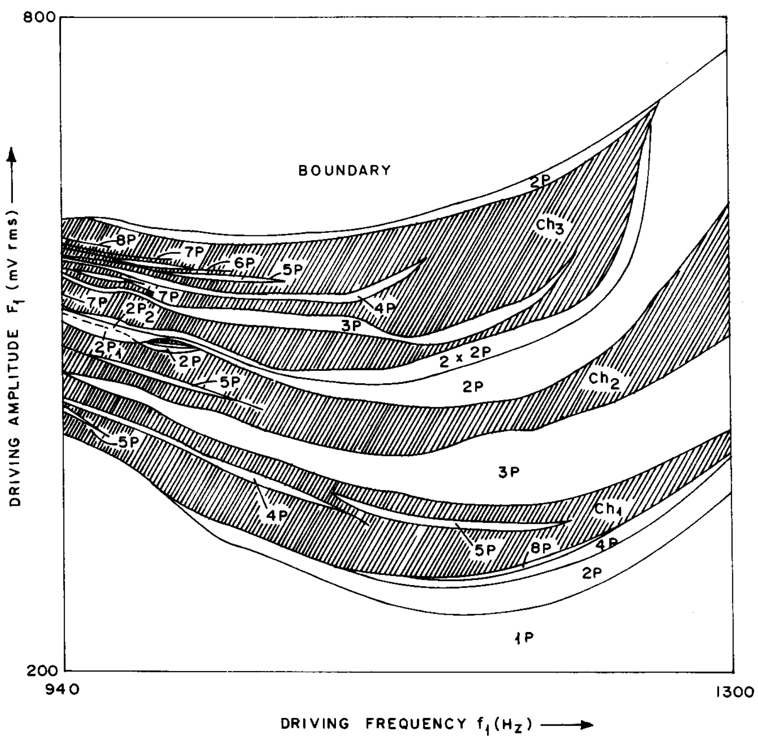


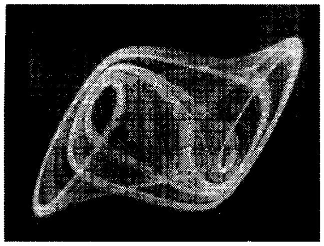
Fig. 2. Profile of bifurcation diagram in $F_1 - f_1$ plane. Shaded regions indicate chaos and numbers denote period of windows.

devil's staircase structures, and period-adding sequences have been observed for this choice of circuit parameters [3].

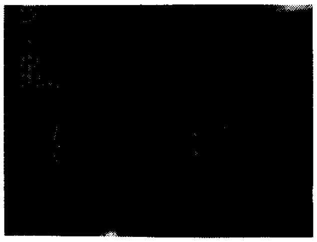
III. Controlling of Chaos

From a practical point of view, it is often desired to control chaotic orbits to an intended periodic orbit and much work has been done in this context [9]–[11]. In this connection, we have now considered one of the easily implementable methods of controlling chaos in the driven Chua's circuit by considering the influence of one more periodic force. Already this method of controlling of chaos has been reported by other authors in different systems numerically [9]–[11].

Presently, we consider the effect of the second periodic signal $F_2 \sin \omega_2 t$ to the circuit of Fig. 1 so that the system becomes a quasi-periodically driven one (F_1, $F_2 \neq 0$). We

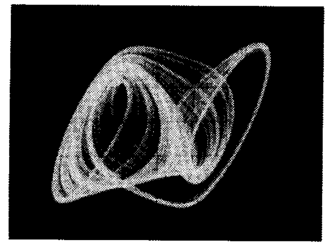


(a)

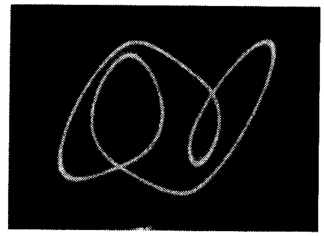


(b)

Fig. 3. (a) Typical trajectory plot in the $V_{C1} - V_{C2}$ plane of a chaotic attractor ($F_1 = 500$ mV and $f_1 = 1200$ Hz); (b) Poincaré map of (a).



(a)



(b)

Fig. 4. Controlling of chaos in the Ch_2 region of Fig. 2 ($F_1 = 584.7$ mV, $f_1 = 1200$ Hz, and $f_2 = 10$ kHz): (a) Double-band chaotic attractor in $V_{C1} - V_{C2}$ plane ($F_2 = 0$); (b) period-3 attractor ($F_2 = 212.2$ mV).

fix the amplitude F_1 and frequency f_1 at a value for which chaos is initially observed (with $F_2 = 0$), then the amplitude F_2 of the second periodic force is slowly increased from zero ($F_2 > 0$, $f_2 (= \omega_2/2\pi) = 10$kHz). Due to the effect of F_2, we observe a remarkable suppression of chaos to take place giving rise to ordered motions. For example, when $F_2 = 0$, a chaotic attractor of Fig. 4(a) is observed initially for $F_1 = 584.7$mV ($f_1 = 1200$ Hz) in the Ch_2 region of Fig. 2. Then, as F_2 is slowly increased to a value of 212.2 mV ($f_2 = 10$ kHz), the chaotic attractor of Fig. 4(a) is controlled to a period-3 window of Fig. 4(b). We have performed our experiments pertaining to all the three chaotic regimes (Ch_1, Ch_2, and Ch_3) and in general due to the effect of the second periodic perturbation F_2, the circuit's chaotic behavior is controlled to the nearby periodic attractors [4].

IV. Synchronization of Chaos

Recently, it was shown that it is possible to construct a set of chaotic systems so that their common signals will have identical or synchronized behavior [8], [12]. In this section, we numerically investigate a simplified form of the driven Chua's circuit with a different set of parameters. In the absence of the second periodic signal ($F_2 = 0$), (1) can be recast into the dimensionless form [5]:

$$\dot{x} = \alpha(y - h(x)), \quad \dot{y} = x - y + z, \quad \dot{z} = -\beta y + F_1 \sin\omega_1 t + \xi(t), \quad \cdot\,(\,\cdot = d/dt). \tag{2}$$

Here, $h(x) = M_1 x + 0.5(M_0 - M_1)[|x + 1| - |x - 1|]$ and $\xi(t)$ is an additional Gaussian noise term with standard deviation σ. Equation (2) is dynamically equivalent to (1) but is more convenient since some parameters are normalized. For our present numerical analysis of (2), we fix $\alpha = 7.0$, $\beta = 14.286$, $\omega_1 = 3.0$, $M_0 = -1/7$, and $M_1 = 2/7$. As the amplitude F_1 is varied from zero, the system exhibits the period doubling bifurcations to chaos. Fig. 5a(i) and b(i) represent the chaotic attractor and its corresponding Poincaré map in the x–y plane for $F_1 = 1.5$ and $\sigma = 0.0$. In addition, Fig. 5a(ii) and b(ii) depict the noise influenced chaotic attractor and its Poincaré map, respectively, for $F_1 = 1.5$ and $\sigma = 0.1$.

A. Drive-Response System

In order to investigate the synchronization of chaos in (2), let us now consider the drive-response scenario using the method of Pecora and Carroll [12]. Then the cascading system of equations is represented as

$$\begin{aligned} &\dot{x} = \alpha(y - h(x)), \quad \dot{y} = x - y + z, \\ &\dot{z} = -\beta y + F_1 \sin\omega_1 t + \xi(t), \\ &\dot{y}' = x - y' + z', \quad \dot{z}' = -\beta' y' + F_1' \sin\omega_1' t + \xi(t), \\ &\dot{x}'' = \alpha'(y' - h(x'')). \end{aligned} \tag{3}$$

Now we numerically integrate (3) with parameters $\alpha = \alpha' = 7.0$, $\beta = \beta' = 14.286$, $F_1 = F_1' = 1.5$, $\omega_1 = \omega_1' = 3.0$, and the initial conditions $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.2$, $y'(0) = 0.15$, $z'(0) = 0.22$, and $x''(0) = 0.15$. For this set of parameters (3) shows chaos as in Fig. 5a(i) and b(i). Even though (3) exhibits chaos and the initial conditions are different, the system exhibit synchronization of chaotic behavior in which the variables y', z', and x'' become identical as time evolves to y, z, and x, respectively, after the initial transients die out. Fig. 5c(i) and (ii) depict the synchronization behavior in which the Poincaré map projected along the x–x'' plane in both the absence ($\sigma = 0$) and the presence ($\sigma = 0.1$) of noise terms, respectively.

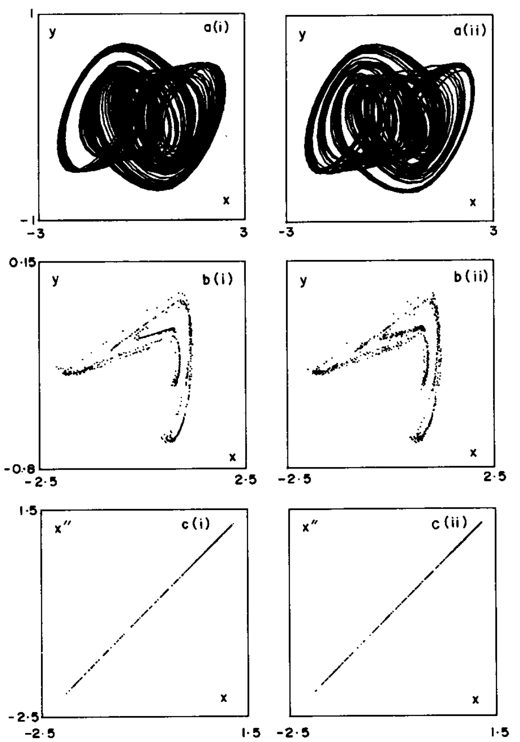


Fig. 5. Chaotic attractor for $F_1 = 1.5$ and $\omega_1 = 3.0$: (i) $\sigma = 0.0$ and (ii) $\sigma = 0.1$; (b) Poincaré map of (a); (c) synchronized motions: Poincaré points are projected in x–x'' plane.

B. Coupled System

Here we have considered two identical driven Chua's circuits mutually coupled by a linear resistor. The state equations are represented as

$$\begin{aligned}
&\dot{x} = \alpha(y - h(x)) - \alpha\epsilon(x - x'), \quad \dot{y} = x - y + z,\\
&\dot{z} = -\beta y + F_1 \sin \omega_1 t,\\
&\dot{x}' = \alpha(y' - h(x')) + \alpha\epsilon(x - x'),\\
&\dot{y}' = x' - y' + z', \quad \dot{z}' = -\beta y' + F_1' \sin \omega_1' t.
\end{aligned} \tag{4}$$

Here ϵ is the coupling parameter. By numerically integrating (4) with parameters $\alpha = 7.0$, $\beta = 14.286$, $F_1 = F_1' = 1.5$, $\omega_1 = \omega_1' = 3.0$, and with initial conditions $x(0) = 0.1$, $y(0) = 0.1$, $z(0) = 0.2$, and $x'(0) = 0.15$, $y'(0) = 0.11$, $z'(0) = 0.22$, then for $\epsilon = 0$, the system depicts unsynchronized motions as shown in Fig. 6(a). But for $\epsilon = 1.0$, synchronized motions are observed as in Fig. 6(b). Similar synchronization of chaotic behavior can also be observed for the y and z coupled systems.

Interestingly, the synchronization of chaotic behavior in cascading systems (3) can be effectively used to transmit signal in a secure way thus opening up new avenues of research in the aspect of *spread-spectrum communications*. Our preliminary numerical investigation reveals that the present system can be used as an ideal model to study secure communication aspects.

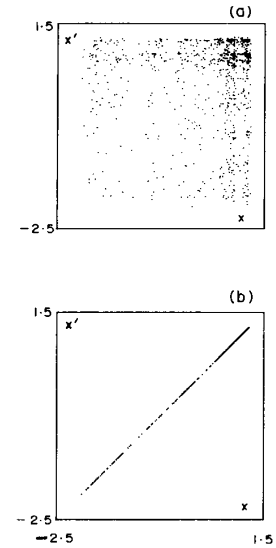


Fig. 6. (a) Unsynchronized motion: Poincaré points in x–x' plane for $\epsilon = 0.0$; (b) synchronized motion: Poincaré points in x–x' plane for $\epsilon = 1.0$.

V. Conclusion

In this paper, we have briefly discussed the chaotic dynamics of the driven Chua's circuit, its rich variety of bifurcations, and chaotic structures and a simple way of controlling chaos in this circuit experimentally. Also, we briefly pointed out the possibility of observing synchronization of chaos numerically. Thus this study reveals that this simple circuit can be effectively utilized to study various features of chaotic dynamics.

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