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CHUA'S CIRCUIT AND THE QUALITATIVE THEORY OF DYNAMICAL SYSTEMS

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Simple electronic oscillators were at the origin of many studies related to the qualitative theory of dynamical systems. Chua's circuit is now playing an equivalent role for the generation and understanding of complex dynamics.

In honour of my friend Leon Chua on his 60th birthday.

1. Oscillating Circuits and the Origin of the Qualitative Theory

In the 19th century, Joseph Fourier wrote: "*The study of Nature is the most productive source of mathematical discoveries. By offering a specific objective, it provides the advantage of excluding vague problems and unwieldy calculations. It is also a means to form the Mathematical Analysis, and isolate the most important aspects to know and to conserve. These fundamental elements are those which appear in all natural effects*".

The important development of the theory of dynamic systems during this century essentially has its origins in the study of the "natural effects" encountered in systems of mechanical, electrical, or electronic engineering, and the rejection of non-essential generalizations. Most of the results obtained in the abstract dynamic systems field have been possible on the foundations of results of the concrete dynamic systems field. It is also worth noting that the majority of scientists (including mathematicians) were not led to their discoveries by a process of deduction from general postulates or general principles, but rather by a thorough examination of properly chosen cases and observation of concrete processes. The generalizations have come later because it is far easier to generalize an established result than to discover a new line of argument.

Since Andronov (1932), three different approaches have traditionally been used for the study of dynamical systems: *Qualitative methods, analytical methods and numerical methods.* To define the "strategy" of *qualitative methods*, one has to note that the solutions of equations of nonlinear dynamic systems are in general nonclassical, transcendental functions of Mathematical Analysis, which are very complex. This "strategy" is of the same type as the one used for the characterization of a function of the complex variable by its singularities: Zeros, poles, essential singularities. Here, the complex transcendental functions are defined by the singularities of continuous (or discrete) dynamic systems such as:

— stationary states which are equilibrium points (fixed points), or periodical solutions (cycles), which can be stable, or unstable;
— trajectories (invariant curves,) passing through saddle singularities of two-dimensional systems;
— stable and unstable manifolds for dimensions greater than two;
— boundary, or separatrix, of the influence domain (domain of attraction, or basin) of a stable (attractive) stationary state;
— homoclinic, and heteroclinic singularities;
— or more complex singularities of fractal, or nonfractal type.

The qualitative methods consider the nature of these singularities in the phase (or state) space, and their evolution when parameters of the system vary, or in the presence of a continuous structure modification of the system (study of the *bifurcation* sets in the parameter space, or in a function space) [Andronov *et al.*, 1966, 1966a, 1967].

In fact, at the beginning qualitative methods developed from the fundamental studies of circuits of radio-engineering. Indeed in 1927, Andronov, the most famous student of Mandelstham, defends his thesis with the topic formulated by Mandelstham *The Poincaré limit cycles and the theory of oscillations*. This thesis is a first-rank contribution for the evolution of the theory of nonlinear oscillations because it opens up new possibilities for application of Poincaré's qualitative theory of differential equations, with great practical significance. With this work, Andronov was the first to see that phenomena of free (or self) oscillations, for example that generated by the Van der Pol oscillator, correspond to limit cycles. It is from the study of oscillators that afterwards, Andronov amplifies his activity with a precise purpose: The development of a *theory of nonlinear oscillations*, in order to make use of mathematical tools common to different scientific disciplines [Andronov *et al.*, 1966].

Andronov and Pontrjagin formulated in 1937 the necessary and sufficient conditions for structural stability of *autonomous two-dimensional systems*. These conditions are: The system only has a finite number of equilibrium points and limit cycles, which are not in a critical case in the Liapunov's sense; no separatrix joins the same, or two distinct saddle points. In this case, it is possible to define, in the parameter space of the system, a set of cells inside of which the same qualitative behavior is preserved [Andronov *et al.*, 1966].

The knowledge of such cells is of first importance for the analysis, and the synthesis of dynamic systems in physics or engineering. On the boundary of a cell, the dynamic system is structurally unstable, and for *autonomous two-dimensional systems* (*two-dimensional vector fields*), *structurally stable systems are dense* in the function space. Until 1966, the conjecture of the extension of this result for higher dimensional systems was generally believed to be true.

Andronov also extended the notion of structural stability for dynamic systems described by:

$$\begin{aligned} dx/dt &= f(x, y)\,, \\ \boldsymbol{\mu} dy/dt &= g(x,y)\,, \quad \boldsymbol{\mu} > 0\,, \end{aligned} \tag{1}$$

where x, y, are vectors, $\boldsymbol{\mu}$ is a "small" parameter vector representing the parasitic elements of the system, $f(x, y)$, and $g(x, y)$ are bounded and continuous in the domain of interest of the phase space. If $\boldsymbol{\mu} = 0$, (1) reduces to a system of lower dimension,

$$dx/dt = f(x, y), \quad g(x, y) = 0. \tag{2}$$

For theoretical, as well as practical purposes, a fundamental problem consists in determining when the "small" terms $\boldsymbol{\mu} dy/dt$, representing the effects of the parasitic elements (small capacitances and inductances in an electrical system, small damping and inertia in a mechanical one) are negligible. In other words, when is the motion described by (1) sufficiently close to the motion described by (2), so that it can be represented by the solution of (2) defined for a lower dimension?

It is interesting to note that the formulation of this important problem has its origin in a discussion (1929) between Andronov and Mandelstham, related to the *one time-constant electronic multivibrator*. Without considering the parasitic elements, such as parasitic capacitances, and inductances, the multivibrator is nominally described by a first-order (one-dimensional) autonomous differential equation, such as (2) where x is now a scalar (voltage). If it is required that $y(t)$ be a continuous function of time, then it was shown by Andronov that (2) does not admit any non-constant periodic solution. Such a mathematical result is contrary to physical evidence, because the one time-constant multivibrator is known to oscillate with a periodic waveform. In the Mandelstham—Andronov's discussion of this paradox, the following alternative was formulated: (a) either the nominal model (2) is not appropriate to describe the practical multivibrator, or (b) it is not being interpreted in a physically significant way.

Andronov has shown that either term of the alternative may be used to resolve the paradox, provided the space of the admissible solutions is properly defined. In fact, specifying that the solutions must be continuous and continuously differentiable leads to the conclusion that (2) is inappropriate on physical grounds, because the real multivibrator possesses several small parasitic elements. Then this leads to a model in the form (1), the vector $\boldsymbol{\mu}$ being related to the parasitic elements. However (1) appears as rather unsatisfactory from a practical point of view. Indeed the existence and the stability of the required periodic solution depends not only on the *presence* of parasitic parameters, which are difficult to measure in practice, but also on their *relative magnitudes*. Andronov has shown that the strong dependence on parasitic elements can be alleviated by means of the second term of the alternative. This is made by generalizing the set of admissible solutions, defined now as consisting of piecewise continuous and piecewise differentiable functions. Then the first-order differential equation (2) is supplemented by some ⟪*jump*⟫ conditions (called *Mandelshtam conditions*) permitting the joining of the various pieces of the solution, which can now be periodic. The theory of models having the form (1) associated with the problem of dimension reduction, and that of *relaxation oscillators* began with this study.

2. Chua's Circuit and the Contemporary Qualitative Theory

One of the reasons for the popularity of the Chua's circuit is due to the fact that it can generate a large variety of complex dynamics, and convoluted bifurcations, from a simple model in the form of a *three-dimensional autonomous piecewise linear ordinary differential equation* (*flow*). It concerns a concrete realization (with discrete electronic components, or implemented in a single monolitic chip) while the well-known *Lorenz equation*, which is also a three-dimensional flow, is related to a very rough low-dimensional model of atmospheric phenomena, far from the real complexity of the ⟪*nature*⟫.

As mentioned above, until 1966, an extension of two-dimensional structural stability conditions, for dimensions higher than two, was conjectured. But Smale [1966, 1967] showed that this conjecture is false in general. So, it appears that, with an increase of the system dimension, one has an increase of complexity of the parameter (or function) space. The boundaries of the cells defined in the phase space, as well as in the parameter space, have in general a complex structure, which may be a fractal (self-similarity properties) for n-dimensional vector fields, $n > 2$.

Sufficient conditions of structural stability were formulated by Smale in 1963. A system is structurally stable when the fixed (equilibrium) points and periodic solutions (orbits) are structurally stable and in finite number, when the set of nonwandering points consists of these stationary states only, when all the stable and unstable manifolds intersect transversally. Such systems are now known as *Morse–Smale systems*.

The analysis of *bifurcations, which transform a Morse–Smale system into a system having an enumerable set of periodic orbits*, has been a favorite choice of research topic since 1965. There certainly exists a lot of such bifurcations of different types. Gavrilov, Afraimovitch and Shilnikov have studied some of them which were related to the presence of structurally unstable homoclinic, or heteroclinic curves associated with an equilibrium point, or a periodic orbit for a dimension $m > 3$. Their results have contributed to the study of the popular *Lorenz differential equation* ($m = 3$) by Afraimovitch and Shilnikov [Afraimovitch *et al.*, 1983]. *Chua's circuit* belongs to the class of three-dimensional "continuous" dynamical systems (*flows* with $m = 3$). With respect to other studies it has the advantage of exhibiting "physical" bifurcations which transform a Morse–Smale system into a system having an enumerable set of periodic orbits.

Let us consider this class of three-dimensional "continuous" dynamical systems (*flows*), and two-dimensional diffeomorphisms associated with them from a Poincaré section. Newhouse [1979] formulated a very important theorem stating that in any neighborhood of a C^r-smooth ($r \geq 2$) dynamical system, in the space of discrete dynamical systems (diffeomorphisms), there exist regions for which systems with homoclinic tangencies (then with structurally unstable, or nonrough homoclinic orbits) are dense. Domains having this property are called *Newhouse regions*. This result is completed in [Gonchenko *et al.*, 1993] which asserts that systems with infinitely many homoclinic orbits of any order of tangency, and with infinitely many arbitrarily degenerate periodic orbits, are dense in the Newhouse regions of the space of dynamical systems. This has a considerable consequence:

> *Systems belonging to a Newhouse region are such that a complete study of their dynamics and bifurcations is impossible.*

Then only particular characteristics of such systems can be studied, such as the presence of nontrivial hyperbolic subsets (infinite number of saddle cycles). Let us restrict to a one-parameter family of three-dimensional dynamical systems leading to *Newhouse intervals*, and the associated family of two-dimensional diffeomorphisms (differentiable invertible maps). Then in such intervals there are dense systems with an infinite number of stable cycles (periodic orbits) if the modulus of the product of their multipliers (eigenvalues) is less than one, and with infinitely many totally unstable cycles if this modulus is higher than one [Shilnikov, 1994]. This last result furnishes a theoretical foundation to the fact that many of the attractors studied contain a "large" hyperbolic subset in the presence of a finite or infinite number of stable cycles. Generally such stable cycles have large periods, and narrow "oscillating" tangled basins, which are difficult to put in evidence numerically.

Systems having infinitely many unstable periodic orbits (they are not of Morse–Smale type) give rise either to *strange attractors*, or to *strange repellors*. Strange repellors are at the origin of two phenomena: Either that of *chaotic transient* toward only one attractor for small changes of initial conditions, or that of *fuzzy* (or *fractal*) *boundaries* [Grebogi *et al.*, 1983] separating the basins of several attractors. In fact, a fractal basin boundary also gives rise to chaotic transients, but toward at least two attractors in the presence of very small variations of initial conditions. The structure identification of strange attractors and repellors, and the bifurcations giving rise to such a complex dynamics, constitute one of the most important problem of this time.

Strange attractors are presently distinguished into three principal classes: *Hyperbolic, Lorenz-type*, and *quasi-attractors* [Shilnikov, 1994].

Hyperbolic attractors are the limit sets for which Smale's Axiom A is satisfied, and are structurally stable. Periodic orbits and homoclinic orbits are dense and are of the same saddle type, that is the stable (resp. unstable) manifold of all the trajectories have the same dimension. In particular, this is the case of Anosov systems, and the Smale–Williams solenoid. Till now such attractors have not been found in concrete applications.

Lorenz attractors are not structurally stable, though their homoclinic and heteroclinic orbits are structurally stable (hyperbolic). They are everywhere, and no stable periodic orbits appear under small parameter variations [Afraimovitch *et al.*, 1983] (for more references cf. also [Shilnikov, 1994]).

Both hyperbolic and Lorenz attractors are stochastic, and thus can be characterized from the ergodic theory.

Quasi-attractors (abbreviation, of "quasistochastic attractors" [Afraimovitch & Shilnikov, 1983], for more references cf. also [Shilnikov, 1994]) are not stochastic, and are more complex than the above two attractors. A quasi-attractor is a limit set enclosing periodic orbits of different topological types (for example stable and saddle periodic orbits), structurally unstable orbits. Such a limit set may not be transitive. *Attractors generated by Chua's circuits* [Chua, 1992, 1993], *associated with saddle-focus homoclinic loops are quasi-attractors*. For three-dimensional systems, mathematically such attractors should contain infinitely stable periodic orbits, a finite number of which can only appear numerically due to the finite precision of computer experiments. They coexist with nontrivial hyperbolic sets. Such attractors are encountered in a lot of models, such as the Lorenz system, the *spiral-type and the double-scroll* attractor generated by a Chua's circuit, the Henon map, this for certain domains of the parameter space.

The complexity of quasi-attractor is essentially due to the existence of structurally unstable homoclinic orbits (homoclinic tangencies) not only in the system itself, but also in any system close to it. It results in a sensitivity of the attractor structure with respect to small variations of the parameters of the generating dynamical equation, i.e. *quasi attractors are structurally unstable*. Then such systems belong to Newhouse regions with the consequences given above.

In the n-dimensional case, $n > 3$, the situation becomes more complex and the first results (in particular a theorem showing that a system can be studied in a manifold of lower dimension) can be found in [Gonchenko *et al.*, 1993b, 1993c].

In addition to its interest in engineering applications, *Chua's circuit* generates a large number of complex fundamental dynamical phenomena. Indeed it is the source of different bifurcations giving rise to chaotic behaviors (period doubling cascade, breakdown of an invariant torus, etc.). The corresponding attractors are related to complex homoclinic heteroclinic structures. One of these attractors, the *double scroll*, characterized by the presence of three equilibrium points of saddle-focus type, arises from two nonsymmetric spiral attractors. It is different from other known attractors of autonomous three-dimensional systems in the sense that it is multistructural.

3. Conclusion

The important book by Madan [1993] collects many contributions devoted to applied and theoretical questions related to this circuit, which since this publication has given rise to many new developments. So the *synchronization of chaotic signals* generated by Chua's circuit leads to an increasing number of publications, with applications to secure communications [Lozi & Chua, 1993]. Moreover, a wide field of research is beginning to be opened through the use of a two- and three-dimensional grid of resistively coupled Chua's circuits. From such networks, waves and spatiotemporal chaos can be put in evidence with *travelling*, *spiral*, *target*, *scroll waves* [Chua & Pivka, 1995]. Here Chua's circuit is used as the basic cell in a discrete *cellular neural network* (*CNN*).

The study of *quasi-attractors* (which are generated in particular by Chua's circuit) does nothing but begin, and so gives a wide field of research. Such attractors cannot be made structurally stable via any finite parameter unfolding of the corresponding system. Arbitrarily small variations of parameters can lead to significant changes of the attractor structure. This results in the impossibility of attaining a complete description of their dynamics and their bifurcation space. Even for three-dimensional flows the results are not complete. A fortiori the extension to higher dimensional cases is a source of open problems for the future, because it is not trivial and provides the occasion to consider new dynamical phenomena [Shilnikov, 1994]. Chains of *Chua's* circuits may furnish a solution to such a matter. Nevertheless, a complete study of such processes being impossible, future research will only be concerned with some specific and typical properties of systems generating quasi-attractors. Related to the above question is the problem of the formulation of a *good model* [Gonchenko *et al.*, 1992], which has a sufficient number of parameters to analyze all possible bifurcations of the steady states, homoclinic and heteroclinic structures, etc. Applied aspects of quasi-attractors are mentioned in [Shilnikov, 1994]. They are concerned with the development of associative memories, and an approach for understanding the memory mechanisms. As indicated in Sec. 1 simple electronic oscillators originated many studies related to the qualitative theory of dynamical systems. It appears that Chua's circuit is now playing an equivalent role for the generation and understanding of complex dynamics, in relation with many applications.

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