A UNIVERSAL CIRCUIT FOR STUDYING
CHAOS IN CHUA'S CIRCUIT FAMILY

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ABSTRACT

In this paper we present a canonical piecewise-linear circuit capable of realizing *every* member of the Chua's circuit family. It contains only six 2-terminal elements: five of them are linear resistors, capacitors and inductors and only one element is a three-segment piecewise-linear resistor. It is canonical in the sense that: (1) It can exhibit *all* possible phenomena associated with any three-region symmetric piecewise-linear continuous vector fields; (2) It contains the *minimum* number of circuit elements needed for such a circuit. Using this circuit, we have found many new chaotic attractors which have not been observed before. Among them, we report only special example here: a non-fractal chaotic attractor.

1 Introduction

Among general piecewise-linear systems, the class of three-region symmetric (with respect to the origin) piecewise-linear continuous vector fields (henceforth denoted by **L**) is of particular interest and importance. It is proved in [1] and [2] that the dynamical behavior of any vector field ξ in **L** is completely determined by its 6 eigenvalues.

Therefore, if we can build a piecewise-linear circuit whose natural frequencies are equal to an arbitrarily prescribed set of eigenvalues, we can derive all possible phenomena in **L** by analyzing this one circuit alone. Such an attempt for the most general class of Chua's circuit, called the Chua's circuit family, has been mentioned in [3], but no such circuit has been reported to date.

In this paper we present a canonical realization of the Chua's circuit family. It contains the minimum number of circuit elements needed to generate all possible phenomena in any 3-dimensional, 3-region, continuous and symmetric piecewise-linear vector fields. In *Section 2* we give this circuit and the explicit formulas for calculating all parameters of this circuit from a given set of 6 eigenvalues.

We have observed many qualitatively different chaotic attractors from this circuit[4]. In Section 3 we report only one interesting example: a non-fractal chaotic attractor. We also introduce a special 2-dimensional surface called *folded strip* on which a trajectory can travel perpetually and non-periodically without intersecting itself. The geometric structure of the folded strip explains the mechanism of the non-fractal chaotic attractor.

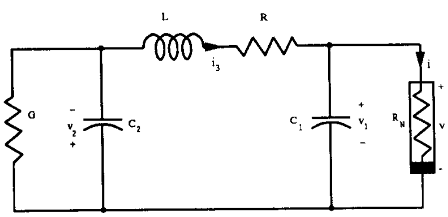

Fig.1(a)

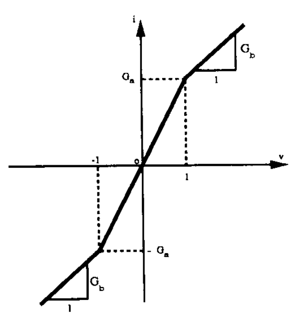

Fig. 1(b)

Fig.1(a) The canonical realization of the Chua's circuit family.
(b) The v-i characteristic of the nonlinear resistor G_N.

2 The canonical piecewise-linear circuit

In this Section we present a *universal* piecewise-linear circuit for realizing *any eigenvalue pattern* associated with any vector field in **L**.

Firstly we have to decide what is the minimum number of elements such a circuit needs. Since our objective is a 3-dimensional 3-region symmetric piecewise-linear continuous vector field, the circuit under consideration is allowed to have only one nonlinear resistor whose v-i characteristic is 3-segment piecewise-linear and symmetric with respect to the origin. The circuit must have three dynamic elements (capacitors and/or inductors) since the system is of third order. The rest are all linear resistors. Let us investigate next how many linear resistors are needed in general.

A linear autonomous R-C circuit has two circuit elements and has only one natural frequency $\mu = 1/RC$. If we increase C to αC and decrease R to R/α , the natural frequency of the circuit will remain unchanged. Therefore, to produce a natural frequency μ , we can assign an arbitrary value to C or R (e.g. let $C = 1$) and find the value for the other parameter.

This means that in a circuit having n parameters, the "degree of freedom" is only $n-1$.

Consider the class of 3-dimensional, 3-region, and symmetric (with respect to the origin) piecewise-linear continuous vector fields. The eigenvalues in the inner region D_0 are denoted by μ_1, μ_2, and μ_3. The eigenvalues in the two outer regions D_{+1} and D_{-1} are equal, since the vector field is symmetric with respect to the origin. We denote them by ν_1, ν_2, and ν_3. Some of the μ's and the ν's may be complex conjugate numbers. In order to avoid complex numbers, let us define

$$p_1 = \mu_1+\mu_2+\mu_3, \quad p_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, \quad p_3 = \mu_1\mu_2\mu_3$$

$$q_1 = \nu_1+\nu_2+\nu_3, \quad q_2 = \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1, \quad q_3 = \nu_1\nu_2\nu_3$$

Since we have six eigenvalues in our problem, we need at least seven parameters. Therefore, besides three dynamic elements and one nonlinear resistor (with two slopes in different regions counted as two circuit parameters), we need at least two linear resistors to build a canonical circuit.

Of course, not every circuit containing that many elements will qualify as a canonical circuit. Our canonical circuit is shown in Fig.1(a).

The state equations of this circuit are:

$$\begin{aligned} \frac{dv_1}{dt} &= \frac{1}{C_1}[-f(v_1) + i_3] \\ \frac{dv_2}{dt} &= \frac{1}{C_2}(-Gv_2 + i_3) \\ \frac{di_3}{dt} &= \frac{-1}{L}(v_1 + v_2 + Ri_3) \end{aligned} \qquad (1)$$

where

$$f(v) = G_b v + \frac{1}{2}(G_a - G_b)\,(|v+1| - |v-1|) \qquad (2)$$

is the v-i characteristic of the nonlinear resistor shown in Fig.1(b).

In the D_0 region (i.e. $|v_1| \le 1$), the state equation (1) become linear:

$$\begin{bmatrix} \frac{dv_1}{dt} \\ \frac{dv_2}{dt} \\ \frac{di_3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-G_a}{C_1} & 0 & \frac{1}{C_1} \\ 0 & \frac{-G}{C_2} & \frac{1}{C_2} \\ \frac{-1}{L} & \frac{-1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} = \mathbf{M}_0 \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix} \qquad (3)$$

where $\mathbf{M}_0$ is a constant matrix. The characteristic equation of $\mathbf{M}_0$ is:

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$$|s\,1-\mathbf{M}_0| = s^3 + s^2\left[\frac{G_a}{C_1}+\frac{G}{C_2}+\frac{R}{L}\right] + s\left[\frac{GG_a}{C_1C_2}+\frac{G_aR}{LC_1}+\frac{GR}{LC_2}+\frac{1}{LC_1}+\frac{1}{LC_2}\right] + \frac{G+G_a+GG_aR}{LC_1C_2} = 0 \quad (4)$$

On the other hand, since μ_1, μ_2 and μ_3 are the eigenvalues of this system, we have

$$(s-\mu_1)(s-\mu_2)(s-\mu_3) = s^3 - p_1s^2 + p_2s - p_3 = 0 \quad (5)$$

Comparing (4) with (5), we obtain

$$\frac{G_a}{C_1}+\frac{G}{C_2}+\frac{R}{L} = -p_1 \quad (6)$$

$$\frac{GG_a}{C_1C_2}+\frac{G_aR}{LC_1}+\frac{GR}{LC_2}+\frac{1}{LC_1}+\frac{1}{LC_2} = p_2 \quad (7)$$

$$\frac{G+G_a+GG_aR}{LC_1C_2} = -p_3 \quad (8)$$

Similarly, from the equation in the $D_{\pm 1}$ regions (i.e. $|v_1>1$) we obtain

$$\frac{G_b}{C_1}+\frac{G}{C_2}+\frac{R}{L} = -q_1 \quad (9)$$

$$\frac{GG_b}{C_1C_2}+\frac{G_bR}{LC_1}+\frac{GR}{LC_2}+\frac{1}{LC_1}+\frac{1}{LC_2} = q_2 \quad (10)$$

$$\frac{G+G_b+GG_bR}{LC_1C_2} = -q_3 \quad (11)$$

Among the seven parameters we can assign an arbitrary value to any one of them. Let us take C_1 as a constant. After some constructive algebraic manipulation[4], we obtain the solution of (6) to (11) as:

$$G_a = \left\{-p_1+\frac{p_2-q_2}{p_1-q_1}\right\}C_1 \quad (12)$$

$$G_b = \left\{-q_1+\frac{p_2-q_2}{p_1-q_1}\right\}C_1 \quad (13)$$

$$L = \frac{1}{\left[p_2+\left(\frac{p_2-q_2}{p_1-q_1}-p_1\right)\left(\frac{p_2-q_2}{p_1-q_1}\right)-\frac{p_3-q_3}{p_1-q_1}\right]C_1} \quad (14)$$

$$R = -L\left[\frac{p_2-q_2}{p_1-q_1}+k\right] \quad (15)$$

$$C_2 = \frac{1}{L\left[\frac{p_3-q_3}{p_1-q_1}+k\left(k+\frac{p_2-q_2}{p_1-q_1}\right)\right]} \quad (16)$$

$$G = kC_2 \quad (17)$$

where k in (15) to (17) is defined as

$$k = -LC_1\left[p_3+\frac{G_a(p_3-q_3)}{C_1(p_1-q_1)}\right] \quad (18)$$

Equations (12) to (18) are explicit formulas for uniquely calculating all parameters in our canonical circuit from *any given set* of eigenvalues.

3. Example of non-fractal chaotic attractor

For this particular example in Section 3, the parameter values are:

$$C_1 = 1.0,\ C_2 = -15.6,\ G = -6.42,$$
$$G_a = 4.13,\ G_b = 0.906,\ L = 0.42,\ R = -0.555 \quad (19)$$

By plotting the trajectory of Eq.(1) with parameter values (19), we get an attractor as shown in Fig.2. The software we use is INSITE[5][6].

We have calculated its Lyapunov exponents. The results listed below are almost independent from initial conditions used for the algorithm.

$$l_1 = 0.0345\ ,\quad l_2 = -0.0000751\ ,\quad l_3 = -0.755 \quad (20)$$

Therefore, the Lyapunov exponents indicate that this attractor is chaotic.

As seen from Fig.2, the attractor looks like a strip. To explore its geometrical structure, we plotted its Poincare cross-sections at different positions. We found that the cross-sections consist of continuous curves. Therefore our conclusion is that

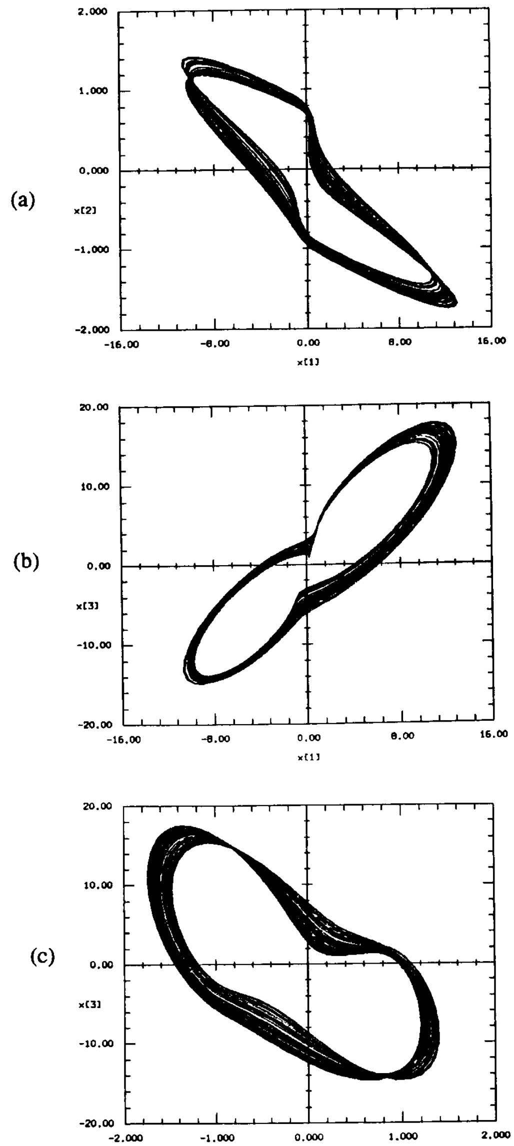

Fig.2 Trajectories of the system (1) with parameter values given in (19). (a) Projection on $v_1 - v_2$ plane. (b) Projection on $v_1 - i_3$ plane. (c) Projection on $v_2 - i_3$ plane. The trajectory travels clockwisely in (a) and (b), counert-clockwisely in (c).

this attractor is located on a 2-dimensional surface and has no fractal structure.

It is well known that the most complicated attractors existing in any 2-dimensional autonomous systems can only be limit cycles. In other words, any attractors on a 2-dimensional plane can only be either equilibrium points or limit cycles, because the trajectory can never intersect itself on a plane, except at equilibrium points.

Now a problem arises: how to explain the attractor we discovered? The answer is: there is no contradiction, because the attractor is located on a 2-dimensional *surface,* not on a *plane*.

Here we introduce a 2-dimensional surface with special geometrical structure and prove that a trajectory on it can be chaotic.

Let us start with a long strip of paper(with no thickness), as shown in Fig.3(a). Imagine we band it around and paste the line AC to the line GH. We then get a *flat closed-strip*. When a trajectory travels along the closed-strip, the only possible attractors are limit cycles. Next imagine we twist the plain strip in Fig.3(a) by 180 degrees and paste the line AC up side down to the line HG (i.e. paste the point A to the point H and paste the point C to the point G). We then get a *Möbius strip*. Although the Möbius strip seems more complicated then the flat closed-strip, still, the only possible attractors on it are limit cycles.

Now let us imagine to fold the long strip in Fig.3(a) along the line BE and paste the rectangular ADEB to the rectangular CFEB. The result is a spoon-like shape called *folded junction,* as shown in Fig.3(b). Then imagine to stretch the whole strip continuously (as if it is a rubber band) and band it around. Finally, paste the line AB to the line GH, as shown in Fig.3(c). We call the resulted shape a *folded strip*. It is a 2-dimensional surface. The crucial part ot if is around the line DE. At this junction, two surfaces *tangential* to each other merge into one. Everywhere else of the folded strip is locally homeomorphic to a 2-dimensional plane.

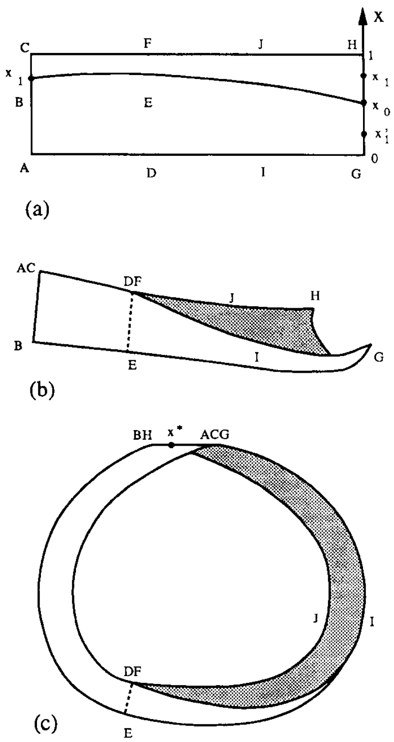

Fig.3(a) A flat strip; (b) A folded junction; (c) A folded strip.

By using a unimodal one-dimensional map we have proved that trajectories on a folded strip can be chaotic[7].

If we look at a sequence of consecutive Poincare cross-sections of the attractor, we can find the folded junction structure. Here we give four consecutive Poincare cross-sections. Figure 4 shows the Poincare cross-sections at $v_2 = 0.9$, 0.6, 0.3 and 0, respectively. Each Poincare cross-section has two branches. Here we plot only one branch associated with the folded junction structure.

The trajectory travels clockwisely on the $v_1 - v_2$ and $v_1 - i_3$ projections and counter-clockwisely on the $v_2 - i_3$ projection. In Fig.4(a)(i.e. $v_2 = 0.9$), the cross-section has just started to fold. In Fig.4(b)(i.e. $v_2 = 0.6$), the cross-section is folded further to form an acute angle. In Fig.4(c)(i.e. $v_2 = 0.3$), the cross-section is almost completely folded. If we look at it

(a) At $v_2 = 0.9$ plane;

(b) At $v_2 = 0.6$ plane;

(c) At $v_2 = 0.3$ plane;

(d) At $v_2 = 0$ plane;

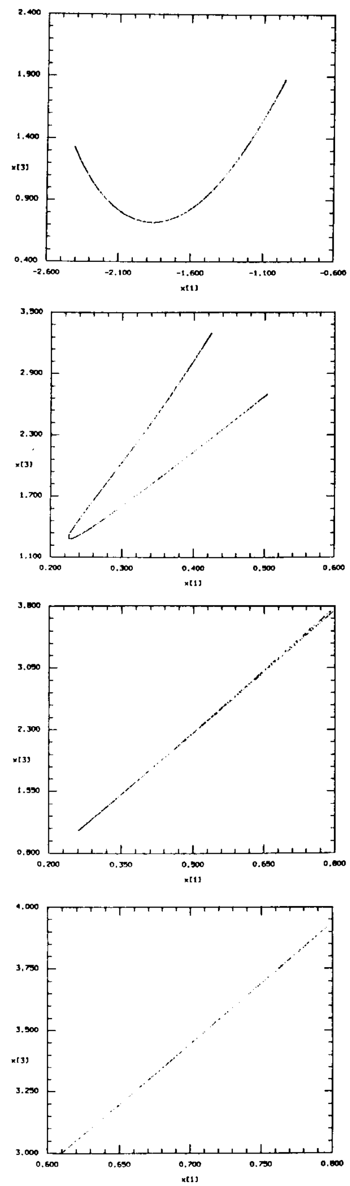

Fig.4 A sequence of successive Poincare cross-sections showing the folded junction structure of the attractor.

carefully, we can still find it is "open" at the upper-right corner of the picture and therefore there is still a small angle. Finally, in Fig.4(d)(i.e. $v = 0$), it has been completely folded. No matter how we zoom it, it is still a line and not an angle. Therefore, this attractor really has the structure of a folded-strip. This explains the mechanism of the non-fractal chaotic attractor quite well.

The fractal dimension is thought as a characterizing feature of chaotic attractors. There are many definitions for different dimensions. In practice, the so-called *Lyapunov dimension* is widely used because it is related to Lyapunov exponents by a simple formula. Let $l_1 \geq \cdots \geq l_n$ be the Lyapunov exponents of an attractor of a continuous-time dynamical system. Let j be the largest integer such that $l_1 + \cdots + l_j \geq 0$. The Lyapunov dimension d_l is defined as

$$d_l = j + \frac{l_1 + \cdots + l_j}{|l_{j+1}|} \quad (21)$$

In case of 3-dimensional vector space, when the three Lyapunov exponents l_1, l_2 and l_3 satisfy

$$l_1 \geq l_2 \geq 0 > l_3 \quad (22)$$

the formula for calculating Lyapunov dimension d_l is

$$d_l = 2 + \frac{l_1 + l_2}{|\, l_3 \,|} \quad (23)$$

Yorke and others classified different definitions of dimension into two large categories: metric dimensions and frequency dependent dimensions. All metric dimensions tend to yield the same value, which is called the *fractal dimension* and denoted by d_F. Similarly, all probabilistic dimensions tend to yield the same value, which is called the dimension of the *natural measure* and denoted by d_μ. Typically, $d_\mu < d_F$. Formula (21) was originally given by Yorke and Kaplan and they conjectured that Lyapunov dimension d_l equates to the dimension of the natural measure d_μ and is a *lower* bound on the fractal dimension d_F ([8][9]).

According to (23), the Lyapunov dimension of this attractor is

$$d_l = 2 + \frac{0.0345}{0.755} = 2.0457 \quad (24)$$

The capacity dimension, which is one of the metric dimensions, can be calculated by

$$d_{cap} = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \quad (25)$$

where ε is the diameter of a small volume element(sphere, cube, etc.) and $N(\varepsilon)$ is the minimum number of such volume elements needed to cover an attractor. Since our attractor is located on a 2-dimensional surface, its capacity dimension d_{cap} is at most 2. From Fig.4 we can see the attractor's cross-sections are continuous curves, which indicate $d_{cap} = 2$ for this attractor.

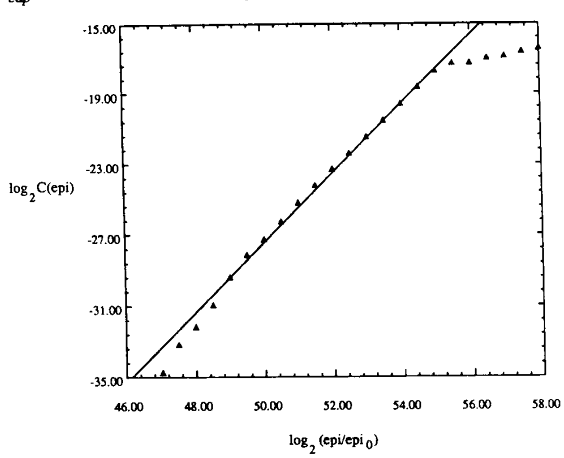

Fig.5 The correlation dimension of the attractor.

We can also calculate the correlation dimension for this attractor. The correlation dimension d_{cor} is defined by

$$d_{cor} = \lim_{\varepsilon \to 0} \ln \sum_{i=1}^{N(\varepsilon)} P_i^2 \,/\, \ln \varepsilon \quad (26)$$

where ε and $N(\varepsilon)$ have the same meaning as in Eq.(25) and P_i is the relative frequency with which a typical trajectory enters the ith volume element. From its definition we know that d_{cor} belongs to the frequency dependent dimensions. To numerically calculate d_{cor}, one can use the *correlation integral* $C(\varepsilon)$ defined by

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \{ \textit{the number of pairs of points } (x_i, x_j) \;\; \textit{such that } ||x_i - x_j|| < \varepsilon \} \quad (27)$$

where N is the total number of points of a trajectory. It can be shown that ([6])

$$C(\varepsilon) = \sum_{i=1}^{N(\varepsilon)} P_i^2$$

Using data number $N = 50000$, the result of calculation is shown in Fig.5. The horizontal axes is $\log_2(\varepsilon/\varepsilon_0)$, where ε_0 is a small constant in the program. The vertical axes is $\log_2 C(\varepsilon)$. In practice, the $\log_2 C(\varepsilon)$ - $\log_2(\varepsilon/\varepsilon_0)$ curve is nearly a straight line in a portion of the curve and the slope of this line is the correlation dimension([6]). For the straight line shown in Fig.5, we have $d_{cor} = 1.98$.

The relationship between d_{cap} and d_{cor} is

$$d_{cor} \leq d_{cap}$$

and the equality takes place only when the trajectory visits each point on its attractor at a uniform probability (i.e. the trajectory is ergodic). Hence $d_{cor} = 1.98 \approx 2$ is a very good numerical result. It means the attractor is nearly ergodic on the 2-dimensional surface, as can be seen from Fig.2.

Therefore, both theoretical analysis and numerical calculation show that this attractor has $d_{cor} \leq d_{cap} \leq 2$. However, it has a positive Lyapunov exponent and hence its Lyapunov dimension $d_l > 2$. So the attractor we obtained seems to be a counter-example to Kaplan-Yorke conjecture.

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