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Application of the differential geometric method to control a noisy chaotic system via dither smoothing

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Abstract

The differential geometric method essentially requires a smooth field in a system model. It cannot be applied to a chaotic system that has a continuous but undifferentiable nonlinearity, e.g., Chua's circuit. This drawback is removed via dither smoothing techniques; then, the controlled system may work for previously unworkable nonlinearities. © 1998 Elsevier Science B.V.

Keywords: Differential geometric method; Dither; Chua's circuit; Saturation; Equivalent nonlinearity

1. Introduction

The control of a chaotic system by using the differential geometric method (DGM), also known as the state-feedback linearization technique, has received much attention in the past few years [1–4]. The main concept of the approach is to algebraically transform the dynamics of a nonlinear system into a linear one by making an appropriate change of coordinates and by applying a nonlinear state feedback, so that the system can be manipulated by linear control techniques. Compared to the pioneering Ott–Grebogi–Yorke (OGY) type methods [5–7], the DGM has several advantages: it performs the jobs automatically after being designed, works well even when the desired trajectory is outside the strange attractor, has the ability to reject noises [4] and shortens the control time (no need to wait for a long time for the trajectory to close in on the desired orbit), etc. Nevertheless, the DGM also has some disadvantages and drawbacks in its inherence; for instance, a larger control size is required and a sufficiently smooth (i.e. continuous partial derivatives to any order) model is needed to construct a differentiable map with a differentiable inverse known as a diffeomorphism. Unfortunately, some inherent nonlinearities that occur in practice have the forms of continuous but undifferentiable properties, such as relay, saturation, dead zone, backlash, etc. A typical case is the chaotic Chua circuit [8] that has a saturation nonlinearity. Inevitably, for these undifferentiable nonlinearities, the DGM failed.

The control of Chua's circuit has been studied in the stabilization case [9,10] with three control inputs, which results in a specific limit cycle. Moreover, a local linearization (Jacobian linearization) has been applied to a variant of Chua's circuit [11] with some restricted control conditions. Recently, a dither smoothing technique has been proposed [12] to stabilize the system, with an experimental test which shows an asymptotically stable result. On the basis of a full state

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measurement and exact state information (no noise), all the above methods can only control the system into a fixed point or a specific limit cycle in the local sense.

Nonlinear control with the aid of a dither signal is well developed in engineering [13,14]. By an appropriate injection of a high-frequency signal, the undifferentiable nonlinearity can be smoothed, the result is called the equivalent nonlinearity. In general, this equivalent nonlinearity may be still undifferentiable but it can be easily approached by some series, e.g., power series, Fourier series, etc. In view of this fact, a dither signal is proposed to smooth the nonlinearity of Chua's circuit. Then, this equivalent nonlinearity is readily represented by a power series. Consequently, the controller design is based on this modified system (Chua's circuit with dither smoothing and series approximation) that is sufficiently smooth and can be manipulated by the DGM. Moreover, using the techniques in Ref. [4], an estimator is designed based on the linearized system of the DGM. Hence, errors which occur due to the series approximation can be easily diminished by an appropriate series-order selection and loop-gain design. As a result, the noisy Chua circuit is effectively controlled in both regulating and tracking with measuring only a single state.

2. System description

Consider a nonlinear chaotic system in the form

$$\begin{aligned}\dot{\boldsymbol{y}} &= \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{N}(\boldsymbol{y}) + \boldsymbol{G}(\boldsymbol{y})(\boldsymbol{u}+\boldsymbol{d}) \\ &= \boldsymbol{F}(\boldsymbol{y}) + \boldsymbol{G}(\boldsymbol{y})(\boldsymbol{u}+\boldsymbol{d}), \\ \boldsymbol{y}' &= \boldsymbol{H}\boldsymbol{y} + \boldsymbol{n}, \end{aligned} \tag{1}$$

where $\boldsymbol{F}(\cdot) = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{N}(\boldsymbol{y})$ and $\boldsymbol{f}(\cdot) : \mathrm{U} \to \mathbb{R}^n$, $\boldsymbol{G}(\cdot) : \mathrm{U} \to \mathbb{R}^{n\times p}$ are sufficiently smooth (i.e. differentiable a sufficient number of times) on a domain $\mathrm{U} \subset \mathbb{R}^n$, $\boldsymbol{N}(\cdot)$ is a continuous but undifferentiable vector and $\boldsymbol{y}'$ is the output with output matrix $\boldsymbol{H}$. To simplify the undifferentiable nonlinearities, we assumed that the vector $\boldsymbol{N}(\cdot)$ has the form

$$\boldsymbol{N}(\boldsymbol{y}) = [n_1(y_1) \quad n_2(y_2) \quad \dots \quad n_n(y_n)]^{\mathrm{T}} \tag{2}$$

which known as a single-valued nonlinearity [14]. For instance, a class of these nonlinear elements is relay, saturation, dead zone, backlash, hysteresis, etc. Moreover, $\boldsymbol{d}$ (system disturbance, $\subset \mathbb{R}^p$) and $\boldsymbol{n}$ (measurement noise, $\subset \mathbb{R}^n$) are simply supposed to be Gaussian, uncorrelated noise (can be changed). Hence, due to the undifferentiable term $\boldsymbol{N}(\cdot)$, the DGM cannot be applied to the system (1). To smooth the vector of $\boldsymbol{N}(\cdot)$, an external signal was injected such that direct access to the nonlinearities input and output signals is possible. Such a signal is usually referred to as artificial dither [13,14], whose frequency will be much higher than the natural frequency of the controlled system. The dither is required at a high frequency for two reasons. First, to ensure that the amount of perturbation it causes to the desired system outputs, and thus also feeds back to the nonlinearity input, is small. Secondly, to justify the substitution of the equivalent nonlinearity for the original characteristic in transient calculations, the dither frequency should be several times higher than the signal component frequency. Hence, the low frequency component of the input signal $y_i(t)$ is nearly constant compared to the dither signal. Therefore, if the low frequency component of the input signal is considered to be a constant over each signal period T, then the output of the equivalent nonlinearity with a power series expansion can be written as

$$\begin{aligned}\bar{n}_i(y_i) &= \frac{1}{T}\int_0^{\mathrm{T}} n_i(y_i + D(t))\,\mathrm{d}t \\ &\simeq \sum_{j=0}^{\infty} \bar{a}_{ij} y_i^j, \quad i = 1,\dots,n, \end{aligned} \tag{3}$$

where $D(t)$ is the dither signal and $\bar{a}_{ij}$ is the coefficients of the power series. Now, system (1) is modified and becomes

$$\dot{\boldsymbol{y}} = \bar{\boldsymbol{F}}(\boldsymbol{y}) + \boldsymbol{G}(\boldsymbol{y})(\boldsymbol{u}+\boldsymbol{d}), \quad \boldsymbol{y}' = \boldsymbol{H}\boldsymbol{y} + \boldsymbol{n}, \tag{4}$$

with

$$\begin{aligned}\bar{\boldsymbol{F}}_\infty(\boldsymbol{y}) &= \boldsymbol{f}(\boldsymbol{y}) + [\bar{n}_1(y_1) \quad \dots \quad \bar{n}_n(y_1)]^{\mathrm{T}} \\ &\simeq \boldsymbol{f}(\boldsymbol{y}) + \left[\sum_{j=0}^{\infty} \bar{a}_{1j} y_1^j \quad \dots \quad \sum_{j=0}^{\infty} \bar{a}_{nj} y_n^j\right]^{\mathrm{T}}. \end{aligned}$$

Obviously, the system (4) is sufficiently smooth. With the (not unique) change of variable $\boldsymbol{x} = \boldsymbol{T}(\boldsymbol{y})$ [15–17], this system can be converted into the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\beta(\boldsymbol{y})^{-1}[\boldsymbol{u} - \alpha(\boldsymbol{y})] + \boldsymbol{B}\boldsymbol{d}',$$

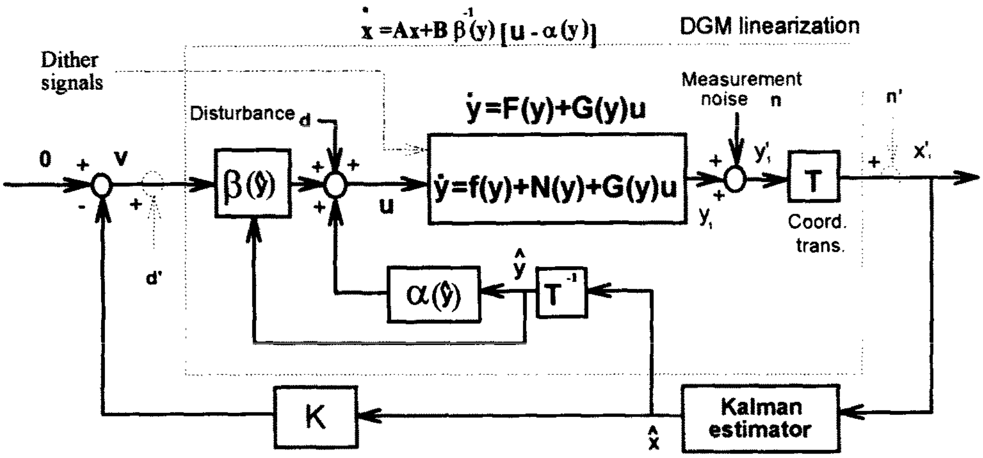

Fig. 1. Schematic diagram of the regulation control of the differential geometric method via dither smoothing.

$$x' = T'(y') = Cx + n', \tag{5}$$

where the pair (A, B) is controllable, and the functions $\alpha : \mathbb{R}^n \to \mathbb{R}^p$ and $\beta : \mathbb{R}^n \to \mathbb{R}^{p\times p}$ are defined in the domain $\mathrm{U} \subset \mathbb{R}^n$ with $\beta(\cdot)$ nonsingular for all $y \in \mathrm{U}$. Moreover, x' is the output vector with output matrix C, T' represents some elements in T, and d', n' are the noises in the transformed coordinates. Hence, the noise $d' = \beta^{-1}(y)d$ is always Gaussian, whereas the elements of the noise n' no more have a Gaussian distribution unless the map $x' = T'(y')$ is linear (more details can be found in Ref. [4]). In Eq. (5) and without the noise, when $u = \alpha(y) + \beta(y)v$ is applied, where v is a linear control input, the known linearized result is

$$\dot{x} = Ax + Bv. \tag{6}$$

For example, in a single-input case, A and B in (6) have the following form,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ \vdots & & & & \ddots & \\ & & & & & 1 \\ 0 & \ldots & & & & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \\ 1 \end{bmatrix}.$$

However, these good results are based on the exact cancellation of the nonlinear terms $\alpha(\cdot)$ and $\beta(\cdot)$. In practice, due to measurement noise, the possible control law, e.g., in the regulating case, is

$$u = \alpha(y') + \beta(y')(-K)T(y'), \tag{7}$$

where K is the regulation gain. On the basis of the techniques in Ref. [4], we design a linear estimator as

$$\dot{\hat{x}} = A\hat{x} + Bv + L(x' - C\hat{x}), \tag{8}$$

where $\hat{x}$ is the estimated state vector and L is the estimator gain. Evidently, the estimator in Eq. (8) is based on the convergence of the control results in Eq. (5), i.e.

$$\dot{x} = Ax + B(v + d'), \quad x' = Cx + n' \tag{9}$$

with the controller

$$u = \alpha(\hat{x}) + \beta(\hat{x})v. \tag{10}$$

The entire control strategies are shown in Fig. 1. Obviously, in order to easily implement the estimator in (8), we expect that the noise (d' and n') in Eq. (9) are Gaussian as well. Thereafter, the choice of relationships in the coordinate transformation is restricted, which means that

(i) T' is a linear map, and

(ii) the pair (A, C) is observable (i.e. through the matrix C the measurement can read out all states in the system).

To this end, we summarize the proposed tactics as follows:

Step 1: Select the dither signals and find the equivalent nonlinearity.

Step 2: Fit the curve by a power series.

Step 3: Design the DGM via dither smoothing and a linear estimator.

3. The control of Chua's circuit

Consider Chua's circuit [8] with the noise described by the following differential equations,

$$\begin{bmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \end{bmatrix} = \begin{bmatrix} r[y_2 - by_1 - n_1(y_1)] \\ y_1 - y_2 + y_3 \\ -sy_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (u+d),$$
$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ 0 \\ 0 \end{bmatrix}, \quad (11)$$

where the function $n_1(y_1)$ is an undifferentiable nonlinearity due to the piecewise-linear resistor, and has the form

$$n_1(y_1) = 0.5(a-b)(|y_1+1| - |y_1-1|), \quad (12)$$

The control input (voltage source) u is added in series with the inductor. In addition, the system only has a single measurement output, y_1. The physical meanings of the states y_1, y_2 and y_3 are the voltage across capacitor 1, the voltage across capacitor 2 and the current through the inductor (more details can be found in Ref. [8]). Let us consider the case where the environmental noises have a Gaussian distribution with zero mean and fixed variance (σ^2), i.e. $n_1 \sim N(0, 0.01^2)$ in measurement and $d \sim N(0, 0.01^2)$ in disturbance. In the sense of the 3σ error, these large noises are equivalent to 30 mV. Here, with the parameters $r = 9$, $s = 100/7$, $a = -0.3$ and $b = 0.4$, the uncontrolled Chua circuit exhibits a chaotic attractor; specifically the double scroll attractor.

3.1. The design procedures

On the basis of the proposed concept, we have the following design steps:

Step 1: Select the dither signals and find the equivalent nonlinearity. A triangular dither signal $D(t)$, with period T and amplitude μ, is selected and described by

$$\begin{aligned} D(t) &= 4\mu/T, & 0 \leqslant t \leqslant T/4, \\ &= 2\mu - 4\mu t/T, & T/4 \leqslant t \leqslant 3T/4, \\ &= 4\mu t/T - 4\mu, & 3T/4 \leqslant t \leqslant T, \end{aligned} \quad (13)$$

Now, let us choose the dither amplitude $\mu = 3$ and frequency = 2000 rad/s, and add these signals in front of the nonlinearity (12). The results of the equivalent nonlinearity are shown in Fig. 2a and formulated as [14]

$$\begin{aligned} \bar n_1(y_1) &= \tfrac{1}{4}(a-b)y_1, & 0 \leqslant y_1 \leqslant 3, \\ &= \tfrac{1}{16}(a-b)[2(4y_1+4+y_1) \\ &\quad -4^2 - 1 - y_1^2], & 3 \leqslant y_1 \leqslant 5, \\ &= a-b, & y_1 \geqslant 5, \\ &= -\bar n_1(-y_1), & y_1 < 0. \end{aligned} \quad (14)$$

Step 2: Fit the curve by a power series. The nonlinearity of Chua's circuit in Eq. (12) is an inverse saturation function. Due to the oddness of (12), the polynomial expansion in y_1 is also odd (i.e. the coefficients of even order terms are zero). Further, to cover all of the possible chaotic region, a local area (which can be easily changed if needed) $|y_1| \leqslant 4$ was considered to fit the nonlinearity. The fitting error, in the least squares sense, is shown in Fig. 2b with an order-5 power series. Accordingly, the polynomial has the form

$$\bar n_1(y_1) \approx \sum_{j=0}^{5} \bar a_{1j} y_1^j = \bar p_5(y_1), \quad (15)$$

where the coefficients of $\bar p_5(y_1)$ are $[0, -1.38 \times 10^{-1}, 0, -5.5718 \times 10^{-4}, 0, 2.8757 \times 10^{-5}]$. Clearly, this nonlinearity can be unknown and the curve fitting acts as the process of system identification in case the input and output are available. In essence, the higher the order in the series approximation, the smaller the error in the control results. Accordingly, from Fig. 2b, we recognize that an order-5 approximation is accurate enough.

Step 3: Design the DGM via dither smoothing and a linear estimator. Now, based on the dither smoothing results (the equivalent nonlinearity) and the order-5

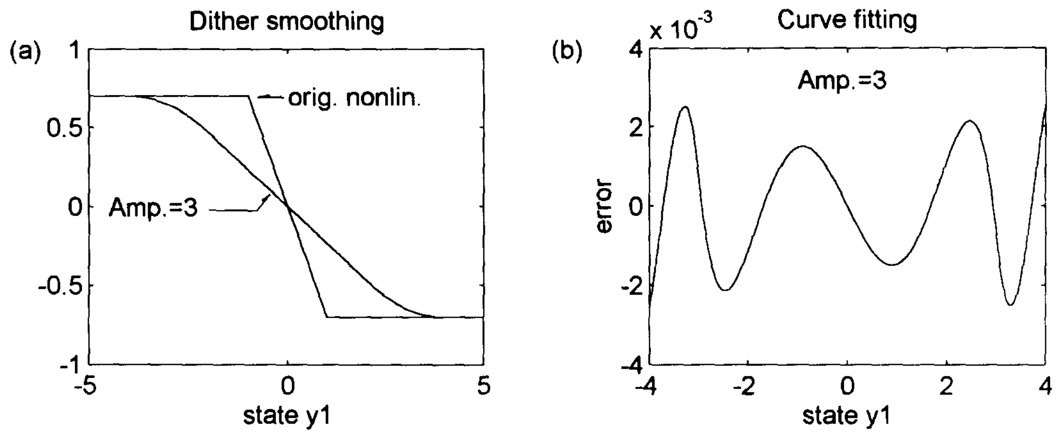

Fig. 2. The equivalent nonlinearity and curve fitting error in Chua's circuit with dither amplitudes $\mu = 3$ and an order-5 series approximation. (a) The equivalent nonlinearity. (b) Curve fitting error.

series approximation, the smoothed Chua circuit becomes

$$\bar{F}_5(y) = \begin{bmatrix} r(y_2 - by_1) \\ y_1 - y_2 + y_3 \\ -sy_2 \end{bmatrix} + \begin{bmatrix} r\sum_{j=0}^{5} \bar{a}_{1j} y_1^j & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$G(y) = [0 \quad 0 \quad 1]^{\mathrm{T}}. \tag{16}$$

Here, the vectors in $\bar{F}_5(\cdot)$ and $G(\cdot)$ are sufficiently smooth. Eq. (16) is (non-uniquely) related to the coordinate transformations $x = T_5(y)$ as (see Appendix A)

$$x_1 = T_{51}(y) = y_1,$$

$$x_2 = T_{52}(y) = ry_2 - rby_1 - rp_5(y_1),$$

$$x_3 = T_{53}(y) = [-rb - r\mathrm{D}p_5(y_1)] \times [ry_2 - rby_1 - rp_5(y_1)] + r(y_1 - y_2 + y_3), \tag{17}$$

where D is a differential operator with respect to the state variable y_1. Eq. (17) shows that the term $p_5(y_1)$ can be substituted by $p_m(y_1)$ which represents the general form of a polynomial of any order. Moreover, the inverse transformations $y = T_5^{-1}(x)$ are obtained as

$$y_1 = T_{51}^{-1}(x) = x_1,$$

$$y_2 = T_{52}^{-1}(x) = bx_1 + p_5(x_1) + (1/r)x_2,$$

$$y_3 = T_{53}^{-1}(x) = (b-1)x_1 + p_5(x_1) + (b + 1 + 1/r)x_2 + x_3. \tag{18}$$

Subsequently, the feedback linearization functions are given by

$$\alpha_5(y) = (\{-r\mathrm{D}^2 p_5(y_1)[ry_2 - rby_1 - rp_5(y_1)] + [rb + r\mathrm{D}p_5(y_1)][rb + \mathrm{D}p_5(y_1)] + r\} \times [ry_2 - rby_1 - rp_5(y_1)] - [rb + r^2\mathrm{D}p_5(y_1) + r](y_1 - y_2 + y_3) - rsy_2)/r, \tag{19}$$

$$\beta_5(y) = 1/r, \tag{20}$$

where D^2 is a twice differential operator with respect to y_1. Similarly, the general form in Eq. (19) is to replace $p_5(y_1)$ by $p_m(y_1)$ (order-m approximation).

Notice that the results in Eqs. (17)–(20) are based on an exact (noise free) and full state measurement. To have noise rejection and state reconstruction, we build a linear estimator as Eq. (8). Here, the transformed relationships follow the restrictions (1) and (2). Because of Eq. (17), we have $x_1' = y_1'$ with $T' = T_1' = 1$ and because of Eq. (9), we have $x_1' = [1\,0\,0]x + n_1'$ with $C = [1\,0\,0]$ and $n_1 = n_1'$. Therefore, Eq. (18) reads

$$\hat{y}_1 = T_{51}^{-1}(\hat{x}) = \hat{x}_1,$$

$$\hat{y}_2 = T_{52}^{-1}(\hat{x}) = b\hat{x}_1 + p_5(\hat{x}_1) + (1/r)\hat{x}_2,$$

$$\hat{y}_3 = T_{53}^{-1}(\hat{x}) = (b-1)\hat{x}_1 + p_5(\hat{x}_1) + (b + 1 + 1/r)\hat{x}_2 + \hat{x}_3.$$

Correspondingly, the feedback cancelling functions $\alpha(\cdot)$ and $\beta(\cdot)$ are obtained. To fulfill the proposed

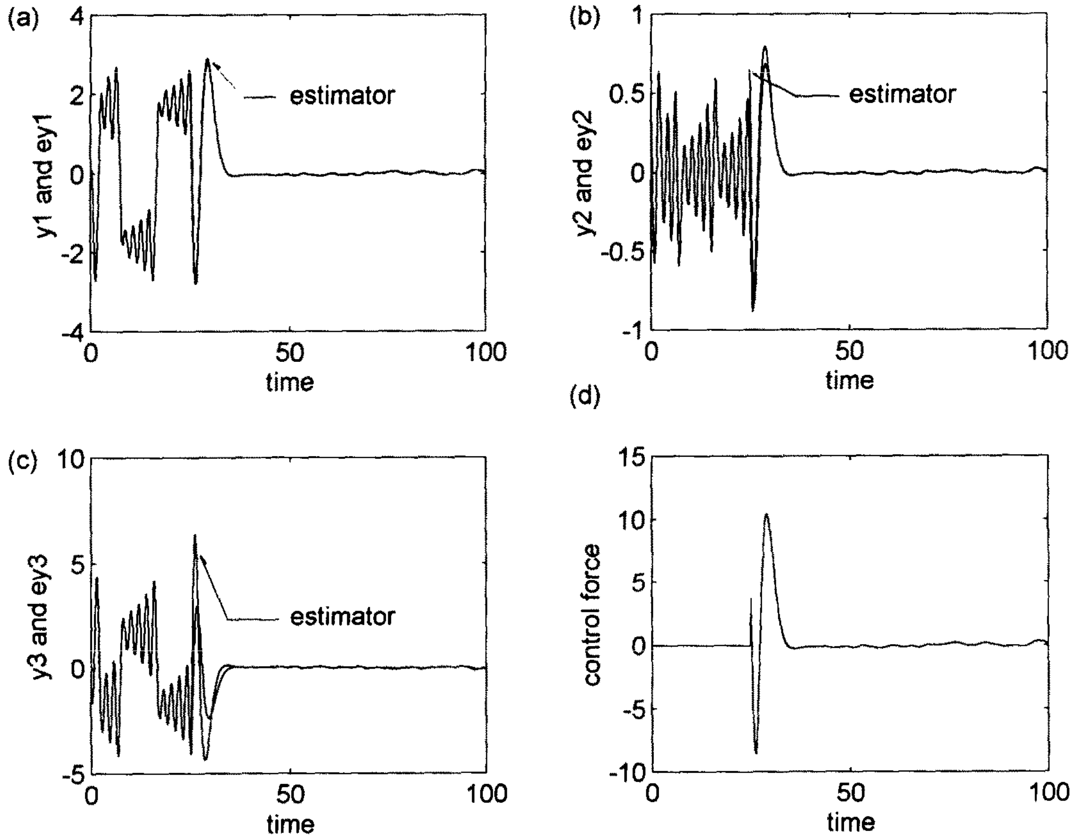

Fig. 3. The chaotic Chua circuit is regulated by the DGM via dither smoothing; the controller is switched on at time $t = 25$ s. (a) The response of state y_1 and estimated state ey_1. (b) The response of state y_2 and estimated state ey_2. (c) The response of state y_3 and estimated state ey_3. (d) The control forces.

design, let the steady Kalman gain be selected based on Eqs. (8) and (9) with $n_1' \sim N(0, 0.01^2)$ and $d' \sim N(0, 0.01^2 r)$. Accordingly, the Kalman gain is

$$L = [2.8845 \quad 4.1602 \quad 3.000]^{\mathrm{T}}.$$

Moreover, the regulation gain in Eq. (10) is chosen as

$$K = [10.000 \quad 18.2674 \quad 11.6848].$$

3.2. Simulations

In simulation, Fig. 3a–3d show the control of a single measurement (y_1) regulation with the controller (including dither and estimator) switched on at $t = 25$ s (which can be arbitrarily changed). The estimator's initial uncertainties are given by the 3σ error in the state y_1 and by 20σ errors in the states y_2 and y_3. With the same conditions in Fig. 3, Fig. 4a–4d shows a single measurement (y_1) tracking results with the controller switched on at $t = 50$ s. From Figs. 3 and 4 it follows that the proposed strategy is feasible and effective.

4. Conclusion

In summary, a smoothing technique in DGM is introduced in this study for the first time. Through dither smoothing and curve fitting processes, the requirements of sufficiently smoothness in the system model are met. Thereafter, inherent nonlinearities that are continuous but undifferentiable can exist in a chaotic system. Moreover, based on the DGM linearized system, an estimator is designed. As a result, the over-

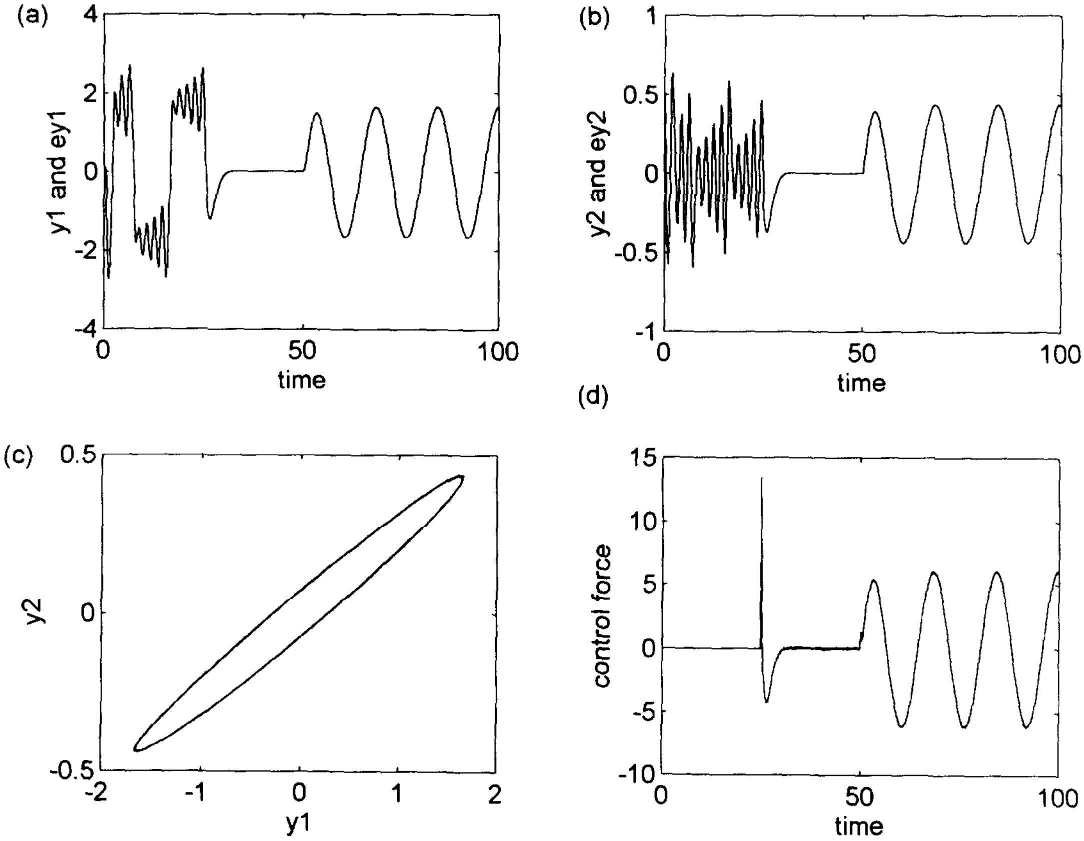

Fig. 4. The chaotic Chua circuit is regulated and tracked by the DGM via dither smoothing; the tracking command is switched on at time $t = 50$ s. (a) The response of state y_1. (b) The response of state y_2. (c) The phase space of states y_1 and y_2 with data recording after $t = 55$ s. (d) The control forces.

all system has the ability to reject noises and to reconstruct the unmeasured states with power-law efficiency. Both regulating and tracking are demonstrated with the controlled results that were previously unworkable.

Appendix A

To derive the relationships of the coordinate transformation, $\boldsymbol{x} = \boldsymbol{T}_m(\boldsymbol{y})$, and feedback cancelling functions, $\alpha_m(\cdot)$ and $\beta_m(\cdot)$, the techniques in Ref. [17] are used to obtain the results in Eqs. (17)-(20). Initially, we have

$$\boldsymbol{F}(\boldsymbol{y}) = \begin{bmatrix} ry_2 - rby_1 - rp_m(y_1) \\ y_1 - y_2 + y_3 \\ -sy_2 \end{bmatrix}, \quad \boldsymbol{G}(\boldsymbol{y}) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

as shown in Eq. (11). Hence, the open-loop Chua circuit is in equilibrium at $\boldsymbol{y} = 0$. We want to find $T_{m1}(\boldsymbol{y})$ satisfying the conditions

$$\frac{\partial T_{m1}}{\partial \boldsymbol{y}} G(\boldsymbol{y}) = 0, \quad \frac{\partial T_{m2}}{\partial \boldsymbol{y}} G(\boldsymbol{y}) = 0, \quad \frac{\partial T_{m3}}{\partial \boldsymbol{y}} G(\boldsymbol{y}) \neq 0 \tag{A.1}$$

with $T_{m1}(0) = 0$, where

$$T_{m2}(\boldsymbol{y}) = \frac{\partial T_{m1}(\boldsymbol{y})}{\partial \boldsymbol{y}} \boldsymbol{F}(\boldsymbol{y}),$$
$$T_{m3}(\boldsymbol{y}) = \frac{\partial T_{m2}(\boldsymbol{y})}{\partial \boldsymbol{y}} \boldsymbol{F}(\boldsymbol{y}). \tag{A.2}$$

From the condition $(\partial T_{m1}/\partial \boldsymbol{y}) G(\boldsymbol{y}) = 0$ in (A.1), we have

$$\frac{\partial T_{m1}}{\partial \boldsymbol{y}} G(\boldsymbol{y}) = \frac{\partial T_{m1}}{\partial y_3} = 0. \tag{A.3}$$

Accordingly, we choose $T_{m1}(\cdot)$ independent of y_3. Further, (A.2) can be written as

$$T_{m2}(y) = \frac{\partial T_{m1}}{\partial y_1}[ry_2 - rby_1 - rp_m(y_1)] + \frac{\partial T_{m1}}{\partial y_2}(y_1 - y_2 + y_3). \qquad (A.4)$$

Again, from the condition $(\partial T_{m2}/\partial y)G(y) = 0$ in (A.1), we have

$$\frac{\partial T_{m2}}{\partial y}G(y) = \frac{\partial T_{m2}}{\partial y_3} = \frac{\partial T_{m1}}{\partial y_2} = 0. \qquad (A.5)$$

We choose $T_{m1}(\cdot)$ independent of y_2. Therefore, in Eq. (A.4), $T_{m2}(\cdot)$ simplifies to

$$T_{m2}(y) = \frac{\partial T_{m1}}{\partial y_1}[ry_2 - rby_1 - rp_m(y_1)]. \qquad (A.6)$$

Moreover, from Eq. (A.2) we have

$$T_{m3}(y) = \frac{\partial T_{m2}}{\partial y_1}[ry_2 - rby_1 - rp_m(y_1)] + r\frac{\partial T_{m1}}{\partial y_1}(y_1 - y_2 + y_3). \qquad (A.7)$$

Hence

$$\frac{\partial T_{m3}}{\partial y}G(y) = \frac{\partial T_{m3}}{\partial y_3} = r \cdot \frac{\partial T_{m1}}{\partial y_1}$$

and the condition $(\partial T_{m3}/\partial y)G(y) \neq 0$ is satisfied globally for all y with any choice of the $T_{m1}(\cdot)$ such that $\partial T_{m1}/\partial y_1 \neq 0$. The simple and unsophisticated choice $T_{m1} = y_1$ satisfies this requirement as well as $T_{m1}(0) = 0$. Subsequently, T_{m2} and T_{m3} are obtained and the change of variables is

$$x_1 = T_{m1}(y) = y_1,$$
$$x_2 = T_{m2}(y) = ry_2 - rby_1 - rp_m(y_1),$$
$$x_3 = T_{m3}(y) = [-rb - r\mathrm{D}p_m(y_1)][ry_2 - rby_1 - rp_m(y_1)] + r(y_1 - y_2 + y_3). \qquad (A.8)$$

The corresponding functions $\alpha_m(\cdot)$ and $\beta_m(\cdot)$ read

$$\beta_m = \frac{1}{\partial T_{m3}/\partial y_3} = \frac{1}{r}, \qquad (A.9)$$

$$\alpha_m = \frac{(\partial T_{m3}/\partial y)F(y)}{(\partial T_{m3}/\partial y)G(y)} = \frac{1}{r}(\{-r\mathrm{D}^2p_m(y_1)[ry_2 - rby_1 - rp_m(y_1)] + [rb + r\mathrm{D}p_m(y_1)][rb + \mathrm{D}p_m(y_1)] + r\} \times [ry_2 - rby_1 - rp_m(y_1)] - [rb + r^2\mathrm{D}p_m(y_1) + r](y_1 - y_2 + y_3) - rsy_2). \qquad (A.10)$$

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