Digital Redesign for Controlling the Chaotic Chua's Circuit

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We apply some successful digital redesign techniques, developed previously for the control of linear systems, to controlling the nonlinear chaotic Chua's circuit. Chua's circuit is a simple autonomous physical device that exhibits very rich and complex nonlinear dynamics of bifurcation and chaos, and is hence very sensitive to digital controls. To apply advanced high-speed computer technology to the implementation, we show how to redesign a good digital controller, based on an existing successful analog controller, for controlling the chaotic trajectories of Chua's circuit, from anywhere within the chaotic attractor to a predesired unstable limit cycle of the circuit.

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I. INTRODUCTION

In the areas of dynamics and control for nonlinear systems, the research on controlling chaos has received increasing attention in the last few years, as can be seen from, for example, the surveys [1, 2, 12, 18].

There are many practical reasons for controlling or ordering chaos. Most of the time, e.g. when chaotic mechanical vibrations occur, chaos is expected to be suppressed. Yet it has been discovered, very recently, that chaos may actually be useful under certain circumstances. In fact, there has been growing interest in meaningful utilization of chaos (see also the aforementioned surveys). Briefly, controlling chaos may be understood as a process, which enhances chaos or increases the amount of chaos when it is beneficial or suppresses it when it is harmful. This process manages the transition between chaos and order and, sometimes, the transition from chaos to different chaos, depending on the situation and purpose.

The methods used to control chaos may be classified into two main categories: feedback and nonfeedback methods. The common feedback methods make use of some essential properties of chaotic systems, e.g. their sensitivity to initial conditions, in order to stabilize some orbits that already exist in the systems. The initial breakthrough of a parameter variation technique was suggested by Ott, Grebogi, and Yorke [13]. They developed a general method for the control of a chaotic system by stabilizing one of its unstable periodic orbits embedded in its attractor via small, time-dependent perturbations of a variable system parameter. From a different perspective, Chen and Dong [3–5] developed some new ideas and formalized some successful techniques for controlling discrete-time and continuous-time chaotic systems to their unstable equilibria or limit cycles using modified conventional engineering feedback controls, based essentially on rigorous Lyapunov arguments.

Controlling chaos in continuous-time systems can be implemented by analog circuits. However, in order to take advantage of the modern high-speed computers and microelectronics, it is more preferable to use digital controllers instead of analog circuits, particularly in aerospace systems and industries [7]. The process of converting an existing continuous-time controller to an equivalent discrete-time controller is called *digital redesign*. Digital redesign is generally very technical, since there are some critical issues such as the sensitive instability problem existing in the redesign when the desired short sampling periods are used in digitization, namely, if the sampling time is too short then it can be very central processing unit (CPU) time consuming and may cause instability in its digital version of the control system [8, 9, 11]. Moreover, fast-rate sampling devices can be very expensive or even physically impossible. For these reasons, digital redesign techniques have never been applied to nonlinear systems, particularly the numerically extremely sensitive chaotic systems.

Digital redesign was first studied in detail by Kuo [9]. He proposed a discrete-state matching method and applied it to a simplified one-axis sky-lab satellite system. Recently, Shieh and his colleagues have thoroughly investigated this type of digital redesign and various types of (sub)optimal digital redesign methods [14, 16–20]. The digital redesign technology has thus been greatly advanced.

In this work, we apply the successful digital redesign techniques to the control of the chaotic Chua's circuit. Chua's circuit is perhaps the simplest autonomous physical device that exhibits very rich and complex nonlinear dynamics such as bifurcations and chaos, and hence has become a prototype of experimental dynamics generator [6, 10]. More specifically, we will show how to redesign a good digital controller, based on an existing successful analog controller, for controlling the chaotic trajectories of Chua's circuit, from anywhere within the chaotic attractor to a predesired unstable limit cycle of the circuit.

The material presented is organized as follows. First, a brief introduction to Chua's circuit is given. Then, an existing continuous-time feedback control approach for controlling the chaotic Chua's circuit is described. In order to apply a digital microprocessor as the controller in implementation, an appropriate digitization of the continuous-time controller is suggested. The digitally controlled Chua's circuit is then studied. The behavior of the digitally controlled Chua's circuit with different sampling periods is compared by computer simulations. Finally, a good digital redesign method for improving the critical control performance of a long sampling-period digital controller for the circuit is introduced and simulated.

We should note that in the current literature, various digital redesign techniques have only been applied to linear systems. Our study carried out in this investigation shows that it can also be applied to some nonlinear, even chaotic, physical systems like Chua's circuit. Our simulation shows that the digitally redesigned controller can provide a satisfactory approximation to the continuous-time controller, and execute the desired control in driving the chaotic trajectory of the circuit to approach its target orbit, an unstable limit cycle of the circuit in this study, with a relatively long sampling time used in the digitization for which standard digital controllers completely fail to work.

II. CHAOTIC CHUA'S CIRCUIT

Chua's circuit is a simple electronic circuit that exhibits a wide variety of nonlinear dynamical phenomena such as bifurcations and chaos. Because of its simplicity and universality, Chua's circuit has

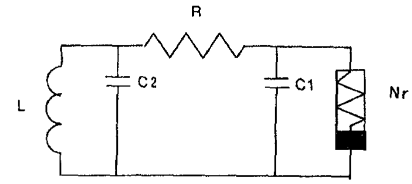

Fig. 1. Diagram of Chua's circuit.

attracted much interest and has become a standard primer on investigations of chaos [10]. The circuit is universal because almost every chaotic and bifurcation phenomenon that has been reported in the literature (e.g., the period-doubling route to chaos, intermittency route to chaos, Hopf-like bifurcations, torus-breakdown route to chaos, chaos in the sense of Shil'nikov's Theorem, etc.) have also been observed from Chua's circuit. It is simple because it contains only one simple nonlinear element (a nonlinear resistor) and four linear elements (two capacitors, one inductor, and a linear resistor). Actual implementation of Chua's circuit can also be found from [10].

Chua's circuit, as shown in Fig. 1, can be described by the following set of dynamical equations:

$$\begin{cases} C_1\dot{v}_{c1} = R(v_{c2} - v_{c1}) - g(v_{c1}) \\ C_2\dot{v}_{c2} = R(v_{c1} - v_{c2}) + i_L \\ L\dot{i}_L = -v_{c2} \end{cases} \tag{1}$$

where i_L is the current through the inductor L, v_{c1} and v_{c2} are the voltages across C_1 and C_2, respectively, and

$$\begin{aligned} g(v_{c1}) &= g(v_{c1}, m_0, m_1) \\ &= m_0 v_{c1} + (m_1 - m_0)(|v_{c1}+1| - |v_{c1}-1|) \end{aligned} \tag{2}$$

with $m_0 \le 0$ and $m_1 \le 0$ being some appropriately constants. Note that, in Fig. 1, N_r denotes the nonlinear resistor described by $g(\cdot)$.

For convenience, and as common practice, we first reformulate the circuit equation (1) to the following dynamically equivalent, but dimensionless, state-space system:

$$\begin{cases} \dot{x}_1 = p(-x_1 + x_2 - f(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -qx_2 \end{cases} \tag{3}$$

where $p = C_2/C_1 > 0$ and $q = C_2/LR^2 > 0$ are the main bifurcation parameters of the circuit, and the nonlinear term represented by the following three-segment piecewise-linear function (see Fig. 2):

$$\begin{aligned} f(x_1) &= m_0' x_1 + 0.5(m_1' - m_0')(|x_1+1| - |x_1-1|) \\ &= \begin{cases} m_0' x_1 + m_1' - m_0' & x_1 \ge 1.0 \\ m_1' x_1 & |x_1| \le 1.0 \\ m_0' x_1 - m_1' + m_0' & x_1 \le -1.0. \end{cases} \end{aligned} \tag{4}$$

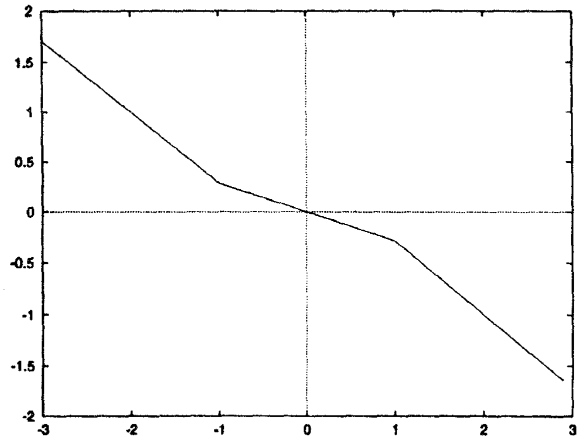

Fig. 2. Graph of three-segment piecewise-linear function.

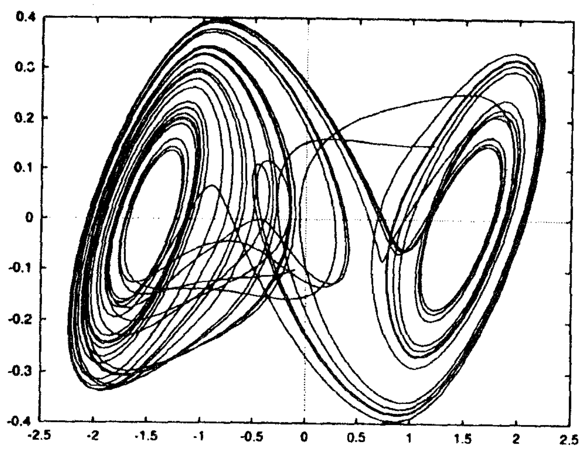

Fig. 3. Chaotic trajectory of Chua's circuit, projected onto x_1-x_2 plane.

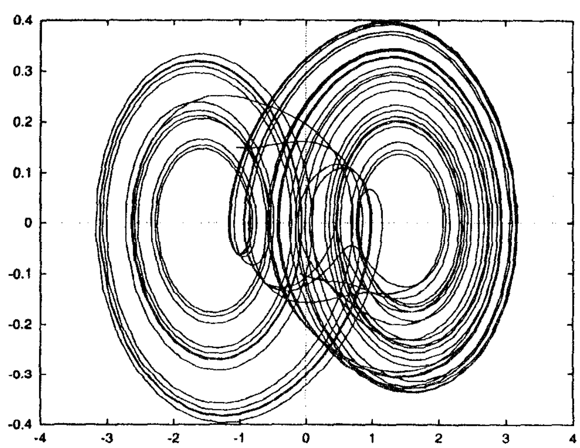

Fig. 4. Chaotic trajectory of Chua's circuit, projected onto x_3-x_2 plane

With $p = 9$, $q = 14\frac{2}{7}$, $m_0' = -\frac{5}{7}$ and $m_1' = -\frac{8}{7}$, the chaotic trajectories of the circuit are shown in Figs. 3–5, where the initial point $(-0.1, -0.1, -0.1)$ was used for the strange attractor [6].

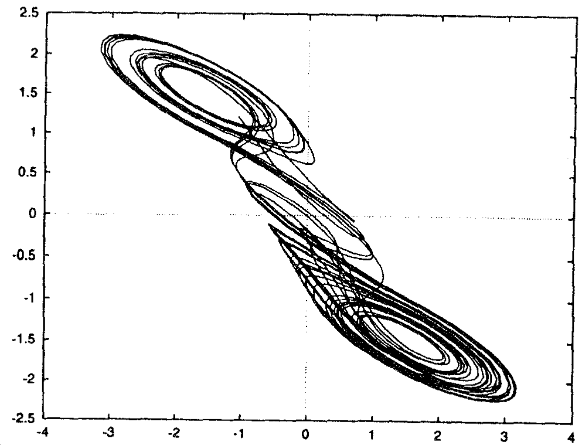

Fig. 5. Chaotic trajectory of Chua's circuit, projected onto x_3-x_1 plane

Since $f(x_1)$ is a three-segment piecewise-linear function, we consider each segment of the function separately and rewrite the circuit equation as a system of three state-space systems in our design to be discussed below. Chua's circuit is thus described by the following three linear state-space systems (which are *continuously connected together*):

$$\dot{x} = Ax + Bu, \tag{5}$$

with the linear segments contained in the B matrix,

$$\dot{x} = \begin{bmatrix} 1.2857 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \qquad |x_1| \leq 1 \tag{6}$$

$$\dot{x} = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} -3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \qquad x_1 \leq -1 \tag{7}$$

$$\dot{x} = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} 3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \qquad x_1 \geq 1. \tag{8}$$

Simulation results of this system of three state-space equations are the same as, at least visually no different from, that shown in Figs. 3–5.

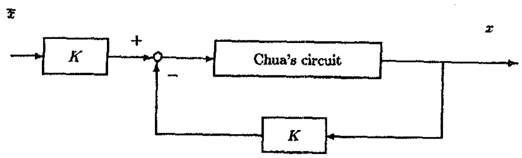

Fig. 6. Feedback control configuration for Chua's circuit.

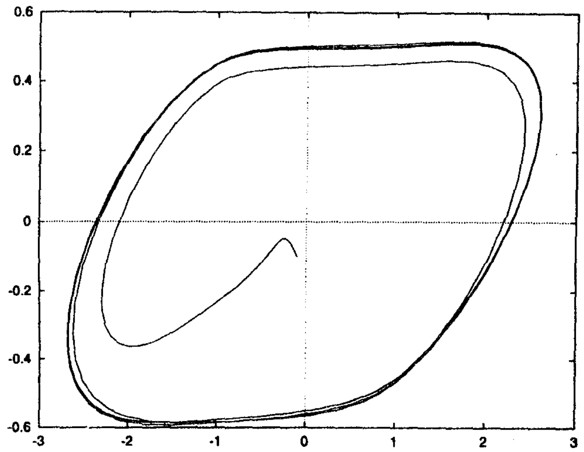

Fig. 7. Trajectory of Chua's circuit, projected onto the x_1-x_2 plane, after feedback control is applied.

III. ANALOG CONTROLLER FOR CHUA'S CIRCUIT

A successful analog feedback controller for controlling the chaotic trajectory of Chua's circuit to its unstable limit cycle was designed by Chen and Dong in [5]. Applying this feedback controller to the circuit, the closed-loop control configuration of the system is shown in Fig. 6. The feedback control that we use is a typical negative state-feedback controller of the form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -K \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ x_3 - \bar{x}_3 \end{bmatrix} = -\begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{bmatrix} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ x_3 - \bar{x}_3 \end{bmatrix} \tag{9}$$

where $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ represents the unstable limit cycle of the circuit, which can be well approximated for the purpose of numerical analysis. For simplicity, we let $k_{11} = 0$ and $k_{33} = 0$, which turns out to be sufficient and works very well. The feedback control after design can be chosen to have $k_{22} = 2.0$ [5].

The trajectory of the circuit, after the feedback control is applied, is shown in Figs. 7–9, where the initial point is $(-0.1, -0.1, -0.1)$ and the designed feedback control is quite effective in directing the trajectory away from the double scroll attractor (see Figs. 3–5) and then driving it to approach the (approximate) unstable limit cycle.

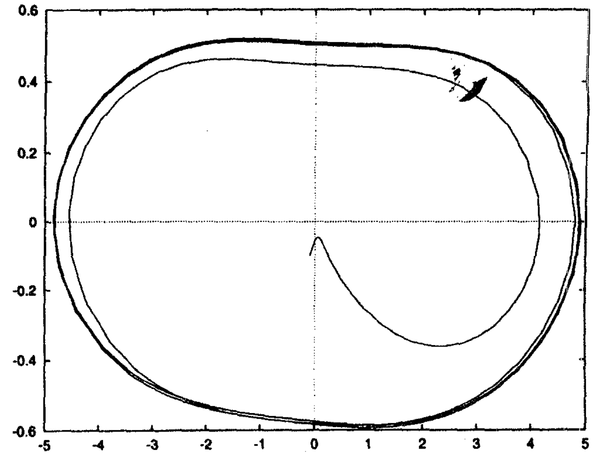

Fig. 8. Trajectory of Chua's circuit, projected onto the x_3-x_2 plane, after feedback control is applied.

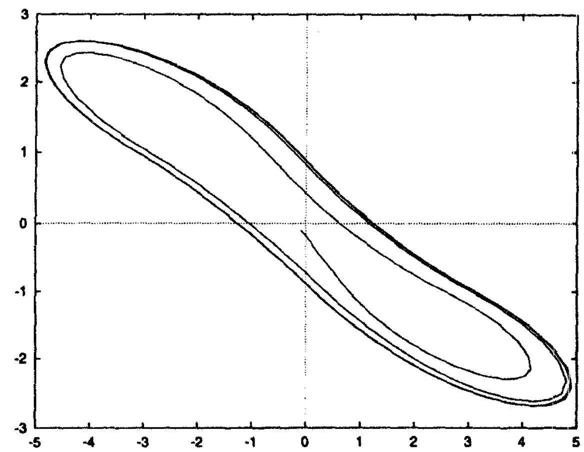

Fig. 9. Trajectory of Chua's circuit, projected onto the x_3-x_1 plane, after feedback control is applied.

Now, consider each segment of the piecewise-linear function separately. The feedback-controlled Chua's circuit can be described by the following system:

$$\dot{x} = Ax + Bu, \tag{10}$$

where

$$u = -K(x - \bar{x}). \tag{11}$$

This system contains three continuously connected linear equations:

$$\dot{x} = \begin{bmatrix} 1.2857 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \quad |x_1| \le 1 \tag{12}$$

where

$$u = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \overline{x}_2 \end{bmatrix},$$

$$\dot{x} = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} -3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \qquad x_1 \leq -1 \tag{13}$$

where

$$u = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \overline{x}_2 \end{bmatrix}, \qquad \text{and}$$

$$\dot{x} = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} x + \begin{bmatrix} 3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} u, \qquad x_1 \geq 1 \tag{14}$$

where

$$u = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \overline{x}_2 \end{bmatrix}.$$

Simulation results of this system of three state-space equations are the same as, at least visually no different from, that shown in Figs. 7–9.

IV. DIGITIZATION OF CHUA'S CIRCUIT

In digital control of continuous-time systems, we need to convert the continuous-time state-space equations to discrete-time state-space equations. Such a conversion can be done by introducing samplers and holding devices into the continuous-time system.

To briefly introduce this notion, consider the following linear continuous-time state-space system:

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \qquad x_c(0) = x_0 \tag{15}$$

where $x_c(t)$ and $u_c(t)$ are an $n \times 1$ state vector and an $m \times 1$ input vector, respectively, and A and B are constant matrices of appropriate dimensions.

The feedback controller depends on the state vector $x_c(t)$ and the $m \times 1$ reference vector $r(t)$, through the relation

$$u_c(t) = E_c r(t) - K_c x_c(t), \tag{16}$$

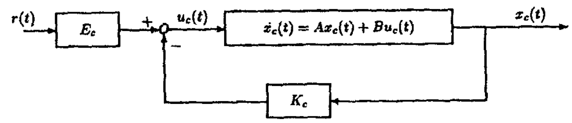

Fig. 10. Continuous-time control system.

where E_c and K_c are an $m \times m$ input matrix and an $m \times n$ feedback matrix, respectively.

The block diagram of this continuous-time system is shown in Fig. 10.

To discretize the continuous-time system, a sampler and a zero-order hold are used. The common industrial practice assumes that the control input $u_c(t)$ is sampled and fed to the zero-order hold, with a sufficiently small sampling period, so that all the components of $u_c(t)$ are constant over the interval between any two consecutive sampling instants. Thus, the sampled and held $u_c(t)$, defined as $\hat{u}_d(t)$, is described by

$$\hat{u}_d(t) = u_c(kT), \qquad \text{for} \quad kT \leq t < kT + T \tag{17}$$

where $T > 0$ is the sampling period, $k = 0, 1, 2, \ldots$.

The solution of (15) is

$$x_c(t) = e^{At}x_c(0) + e^{At}\int_0^t e^{-A\tau}Bu_c(\tau)\,d\tau. \tag{18}$$

Letting $t = kT$ and $t = (k+1)T$, respectively, we obtain

$$x_c(kT) = e^{AkT}x_c(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu_c(\tau)\,d\tau \tag{19}$$

and

$$x_c[(k+1)T] = e^{A(k+1)T}x_c(0) + e^{A(k+1)T} \times \int_0^{(k+1)T} e^{-A\tau}Bu_c(\tau)\,d\tau. \tag{20}$$

From (19) and (20), we have

$$x_c[(k+1)T] = e^{AT}x_c(kT) + e^{A(k+1)T} \times \int_{kT}^{(k+1)T} e^{-A\tau}Bu_c(\tau)\,d\tau. \tag{21}$$

Since, in (17), $u_c(t) = \hat{u}_d(kT)$ for $kT \leq t < kT + T$, we can substitute

$$u_c(\tau) = \hat{u}_d(kT) = \text{constant}$$

in (21) and define $x_c(kT)$ as $\hat{x}_d(kT)$, to obtain its solution as follows:

$$\hat{x}_d[(k+1)T] = e^{AT}\hat{x}_d(kT) + \int_0^T e^{A\lambda}\,d\lambda B\hat{u}_d(kT). \tag{22}$$

If, furthermore, we define

$$G = e^{AT} \tag{23}$$

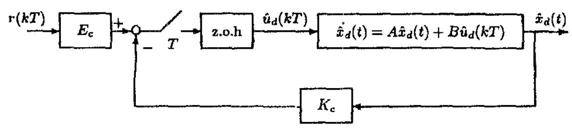

Fig. 11. Digital control system.

and

$$H = \left(\int_0^T e^{A\lambda} \, d\lambda \right) B \tag{24}$$

then (22) becomes

$$\hat{x}_d[(k+1)T] = G\hat{x}_d(kT) + H\hat{u}_d(kT). \tag{25}$$

So far, we have derived the discrete-time state equation that takes values only at $t = kT$, for $k = 0, 1, 2, \ldots$. We have also obtained the discrete-time feedback controller as

$$\hat{u}_d(kT) = -K_c\hat{x}_d(kT) + E_c r(kT) \tag{26}$$

so that the state-space equations of the digitally controlled system becomes

$$\dot{\hat{x}}_d(t) = A\hat{x}_d(t) + B\hat{u}_d(kT) \tag{27}$$

where $kT \le t < (k+1)T$, and

$$\hat{u}_d(kT) = -K_c\hat{x}_d(kT) + E_c r(kT). \tag{28}$$

Here, it is important to note that the matrices A and B are identical to those in system (15).

This digitally controlled system, which is corresponding to the continuous-time system (27), is shown in Fig. 11.

With the above digitization preliminaries, we now return to the feedback-controlled Chua's circuit (12)–(14).

In discretizing the state-space system (12)–(14), using (23)–(24) to obtain the corresponding (25) for the circuit, we have

$$\begin{aligned}\hat{x}_d[(k+1)T] &= G\hat{x}_d(kT) + H\hat{u}_d(kT) \\ &= e^{AT}\hat{x}_d(kT) + \left(\int_0^T e^{A\lambda} \, d\lambda \right) B\hat{u}_d(kT)\end{aligned} \tag{29}$$

where

$$A = \begin{bmatrix} 1.2857 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix}$$

for $|x_1(kT)| \le 1.0$.

Similarly, we can obtain the other two digitized equations with the following:

$$A = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} -3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix}$$

for $x_1(kT) \le -1.0$, and

$$A = \begin{bmatrix} -2.5714 & 9.0 & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -14.2857 & 0.0 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 3.8571 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix}$$

for $x_1(kT) \ge 1.0$.

The initial condition is the same:

$$\hat{x}_d(0) = [-0.1 \quad -0.1 \quad -0.1] \tag{30}$$

and the reference input, on the other hand, is

$$r(kT) = \begin{bmatrix} 1 \\ \overline{x}_2(kT) \end{bmatrix} \tag{31}$$

which is the discrete version of the given unstable limit cycle of the circuit.

As discussed above, the digitized state-feedback controller is

$$\hat{u}_d(kT) = -K_c\hat{x}_d(kT) + E_c r(kT). \tag{32}$$

Referring to (27)–(28), the digitally controlled closed-loop system of the circuit becomes

$$\dot{\hat{x}}_d(t) = \begin{bmatrix} 1.2857 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} \hat{x}_d(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{u}_d(kT) \tag{33}$$

for $|x_1(t)| \le 1.0$, where

$$\hat{u}_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \hat{x}_d(kT) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} r(kT);$$

$$\dot{\hat{x}}_d(t) = \begin{bmatrix} -2.5714 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} \hat{x}_d(t) + \begin{bmatrix} -3.8571 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{u}_d(kT) \tag{34}$$

for $x_1(t) \le -1.0$, where

$$\hat{u}_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \hat{x}_d(kT) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} r(kT);$$

and

$$\dot{\hat{x}}_d(t) = \begin{bmatrix} -2.5714 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} \hat{x}_d(t) + \begin{bmatrix} 3.857 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{u}_d(kT) \tag{35}$$

for $x_1(t) \ge 1.0$, where

$$\hat{u}_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \hat{x}_d(kT) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} r(kT).$$

Using these resulting formulas, we can examine the dynamical behavior of the chaotic Chua's circuit under the digital controller with different sampling periods, which is common practice in industry, and then compare them with the behavior of the original continuous-time circuit under the control of the analog feedback controller. This is further discussed in the next section.

V. COMPUTER SIMULATIONS I: BEFORE REDESIGN

In this section, we examine the dynamical behavior of the chaotic Chua's circuit under the control of the digital controller redesigned above, with different sampling periods, and compare them with the behavior of the original continuous-time circuit under the control of the analog feedback controller developed by Chen and Dong in [5].

Recall that the continuous-time feedback controller designed in [5] works very well for the control purpose. To implement it by a digital computer, however, there is a tradeoff problem between the sampling period and the numerical stability sensitivity in digital control [8, 9, 11]. Let us first use a small sampling period $T = 0.05$ s. Our computer simulations have shown that in this case the trajectories of the digitally controlled chaotic circuit match very closely with those of the continuously controlled circuits, and the results are similar to that shown in Figs. 7–9. Note, however, under laboratory conditions, it is sometimes not so easy to implement a digital control system using such a short sampling period. Generally, the performance of the digital controller is improved with decreasing sampling periods, but it makes the implementation of the digital systems difficult, or impossible, and may cause the well-known sensitive instability problem in digitization [8, 9, 11]. Besides, a high sampling rate requires high speed analog-to-digital (A/D) and digital-to-analog (D/A) converters and a more advanced computer.

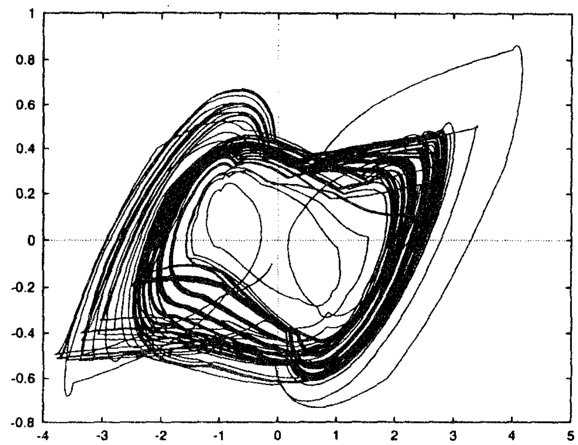

Fig. 12. Digitally controlled trajectory of Chua's circuit, projected onto the x_1-x_2 plane, with the sampling period $T = 0.4$ s.

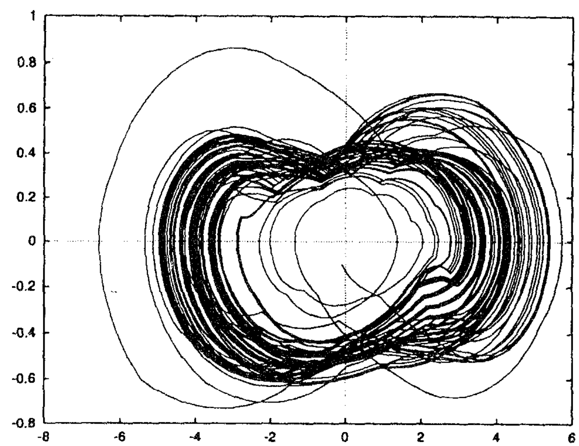

Fig. 13. Digitally controlled trajectory of Chua's circuit, projected onto the x_3-x_2 plane, with the sampling period $T = 0.4$ s.

To further examine the sampling effects in the control of the very sensitive chaotic Chua's circuit, we now suppose that short sampling periods are inconvenient under a certain circumstance, and discuss the behavior of digitally controlled Chua's circuit for a relative long sampling period.

In so doing, we let $T = 0.4$ s, say. Figs. 12–14 show the trajectories of this digitally controlled circuit. It is clear that the trajectory of this digitally controlled system does not approach the target orbit, the unstable limit cycle of the circuit, in any way. As a matter of fact, the trajectory has eventually lost control and then diverges. This shows that the performance of the digitally control system becomes worse when sampling period is getting longer, as expected.

For relatively long sampling periods, how can we resolve this problem, and to redesign our successful analog feedback controller to keep the chaotic trajectory of the circuit on, or nearby, the target orbit? An answer is given in Section VI.

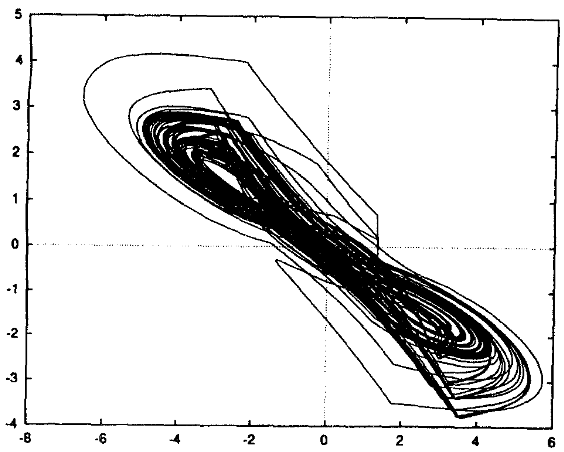

Fig. 14. Digitally controlled trajectory of Chua's circuit, projected onto the x_3-x_2 plane, with the sampling period $T = 0.4$ s.

VI. DIGITAL REDESIGN FOR CONTROLLING CHUA'S CIRCUIT

To replace an already-designed analog controller by a digital one, it is desirable not to carry out a completely new design using digital control theories (sometimes, this is not even possible), but rather, to apply a convenient digital redesign technique that utilizes the original, well-designed analog controller.

Kuo was the first one who considered in detail such a digital redesign methodology [9]. A discrete-state matching method was proposed in [9] to solve the static digital redesign problem for linear systems and then successfully applied it to a simplified one-axis sky-lab satellite system. Kuo's redesign method involves selection of a weighting matrix, and so the result of the state-matching depends heavily on the specific selection of the weighting matrix. To improve this method, Shieh and his colleagues have recently investigated, quite thoroughly, this type of digital redesign and various types of (sub)optimal digital redesign methods [14, 16–20]. The digital redesign technology has thus been greatly advanced.

In this section, we apply Shieh's approach to a redesign of the constant-feedback controller for the control of Chua's circuit, as mentioned above. An equivalent digital control system will first be obtained for the circuit, where a digital control system is said to be equivalent to a continuous-time control system if the responses of the two systems, which may be under the control of two different types of controllers, are closely matched for the same input and the same initial conditions.

To introduce Shieh's digital redesign method, let us consider a linear, time-invariant, controllable, continuous-time system described by

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t), \qquad x_c(0) = x_0 \qquad (36)$$

where $x_c(t)$ and $u_c(t)$ are an $n \times 1$ state vector and an $m \times 1$ control input vector, respectively, and A and B are constant matrices of appropriate dimensions.

Let the state-feedback controller be

$$u_c(t) = -K_c x_c(t) + E_c r(t) \qquad (37)$$

where K_c is an $m \times n$ feedback again matrix, E_c an $m \times m$ forward gain matrix, and $r(t)$ an $m \times 1$ reference input. The resulting closed-loop controlled system is then given by

$$\dot{x}_c(t) = (A - BK_c)x_c(t) + BE_c r(t), \qquad x_c(0) = x_0. \qquad (38)$$

Next, we let the state-space system of a continuous-time system, which contains the same system matrix A and input matrix B but has a different control input, be represented by

$$\dot{x}_d(t) = Ax_d(t) + Bu_d(t), \qquad x_d(0) = x_0 \qquad (39)$$

where $u_d(t)$ is an $m \times 1$ piecewise-constant function:

$$u_d(t) = u_c(kT) \qquad \text{for} \quad kT \le t < kT + T \qquad (40)$$

and $T > 0$ is the sampling period.

Applying a sampler and a zero-order hold to system (39), the solution of the resulting system is obtained as

$$x_d(t) = e^{A(t-kT)}x_d(kT) + \int_{kT}^{t} e^{A(t-\lambda)}B\,d\lambda\, u_d(kT) \qquad \text{for} \quad kT \le t < kT + T. \qquad (41)$$

For $t = kT + T$, the equivalent discrete-time model of the continuous-time system (39) can be written as

$$x_d(kT + T) = Gx_d(kT) + Hu_d(kT), \qquad x_d(0) = x_0 \qquad (42)$$

where

$$G = \exp(AT)$$

and

$$H = \int_0^T e^{A\lambda}B\,d\lambda = [G - I_n]A^{-1}B = \sum_{i=1}^{\infty} \frac{1}{i!}(AT)^{i-1}BT.$$

We then let the discrete-time state-feedback controller of system (39) be

$$u_d(kT) = -K_d x_d(kT) + E_d r(kT) \qquad (43)$$

where K_d is an $m \times n$ digital feedback gain matrix, E_d an $m \times m$ digital forward gain matrix, and $r(kT)$ the $m \times 1$ discrete-time reference input. The resulting

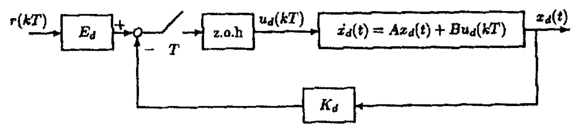

Fig. 15. Configuration of redesigned digital system.

closed-loop control system becomes

$$\dot{x}_d(t) = Ax_d(t) - BK_d x_d(kT) + BE_d r(kT), \quad x_d(0) = x_0 \tag{44}$$

for $kT \leq t < kT + T$.

Now, the digital redesign problem is reduced to finding the digital constant state-feedback gain matrix K_d and forward gain matrix E_d in (43) from the available continuous state-feedback gain K_c and forward gain E_c given in (37), which was obtained from the original design of the analog feedback controller, so that the output trajectories of the digital system (44) closely match that of the analog system (38), for $x_c(0) = x_d(0)$ and for the same reference input.

In Shieh's approach, e.g., the one described in [16], the digitally redesigned state-feedback gain matrix K_d and the forward gain matrix E_d are found to be

$$K_d = (I_m + K_c H^{(v)})^{-1} K_c G^{(v)} \tag{45}$$

and

$$E_d = (I_m + K_c H^{(v)})^{-1} E_c \tag{46}$$

respectively, where

$$G^{(v)} = \exp(Avt)$$

and

$$H^{(v)} = \int_0^T \exp(Av\tau) B \, d\tau = (G^{(v)} - I_n) A^{-1} B$$

in which I_m is the $m \times m$ identity matrix. Here, the choice of the tuning parameter v in (45)–(46) depends upon the specific sampling period $T > 0$ and the desired closeness of the trajectory $x_d(t)$ of the redesigned digital system (44) to the one, $x_c(t)$, of the original continuous-time system (38). The following performance index is suggested in [16], as a design criterion for the selection of the tuning parameter v:

$$J(v) = \sum_{i=1}^{n} \left(\int_0^{t_f} |x_{ci}(t) - x_{di}(t)| \, dt \right)$$

where t_f is the finite terminal time of interest.

The overall redesigned digital system is shown in Fig. 15.

Next, to redesign the analog controller for Chua's circuit, with a relatively long sampling period, $T =$ 0.4 s, for which the original digital controller did not work at all (see Figs. 11–13), we apply Shieh's formulas (45)–(46).

When the sampling period is $T = 0.4$ s, the digitally controlled Chua's circuit is described by the following system of three equations:

$$\dot{x}_d(t) = \begin{bmatrix} 1.2857 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} x_d(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u_d(kT) \tag{47}$$

for $|x_1(t)| \leq 1.0$, where

$$u_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0.4639 & 0.6735 & 0.3560 \end{bmatrix} x_d(kT) + \begin{bmatrix} 1 & 0 \\ 0 & 1.2878 \end{bmatrix} r(kT)$$

$$\dot{x}_d(t) = \begin{bmatrix} -2.5714 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} x_d(t) + \begin{bmatrix} -3.8571 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u_d(kT) \tag{48}$$

for $x_1(t) \leq -1.0$, where

$$u_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0.2058 & 0.4378 & 0.3402 \end{bmatrix} x_d(kT) + \begin{bmatrix} 1 & 0 \\ 0.2015 & 1.3194 \end{bmatrix} r(kT)$$

and

$$\dot{x}_d(t) = \begin{bmatrix} -2.5714 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2857 & 0 \end{bmatrix} x_d(t) + \begin{bmatrix} 3.8571 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u_d(kT) \tag{49}$$

for $x_1(t) \geq 1.0$, where

$$u_d(kT) = -\begin{bmatrix} 0 & 0 & 0 \\ 0.2058 & 0.4378 & 0.3402 \end{bmatrix} x_d(kT) + \begin{bmatrix} 1 & 0 \\ -0.2015 & 1.3194 \end{bmatrix} r(kT).$$

The initial condition used is $x_d(0) =$ $[-0.1, -0.1, -0.1]^T$, and the reference input is

$$r(kT) = \begin{bmatrix} 1 \\ \overline{x}_2(kT) \end{bmatrix}.$$

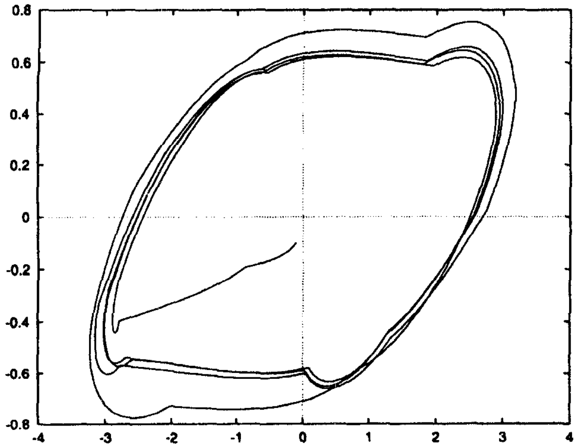

Fig. 16. Trajectory of Chua's circuit under digitally redesigned controller, projected onto x_1-x_2 plane, with sampling period $T = 0.4$ s.

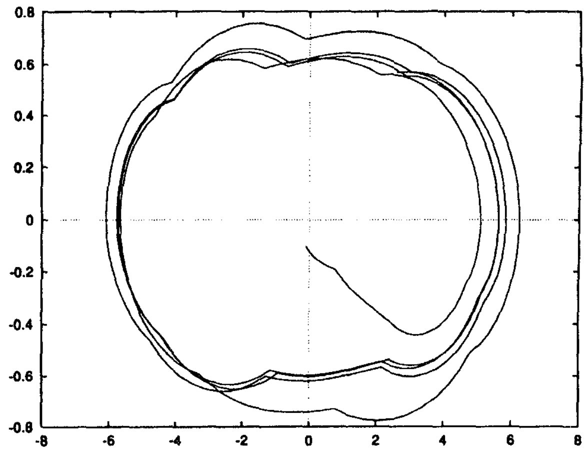

Fig. 17. Trajectory of Chua's circuit under digitally redesigned controller, projected onto x_3-x_2 plane, with sampling period $T = 0.4$ s.

Using the redesign formulas (45)–(46), we choose $v = 0.9$. The feedback and forward gain matrices, K_d and E_d, are obtained, respectively, as follows:

$$K_{d1} = \begin{bmatrix} 0 & 0 & 0 \\ 0.4639 & 0.6735 & 0.3560 \end{bmatrix},$$

$$E_{d1} = \begin{bmatrix} 1 & 0 \\ 0 & 1.2878 \end{bmatrix}$$

$$K_{d2} = \begin{bmatrix} 0 & 0 & 0 \\ 0.2058 & 0.4378 & 0.3402 \end{bmatrix},$$

$$E_{d2} = \begin{bmatrix} 1 & 0 \\ 0.2015 & 1.3194 \end{bmatrix}$$

$$K_{d3} = \begin{bmatrix} 0 & 0 & 0 \\ 0.2058 & 0.4378 & 0.3402 \end{bmatrix},$$

$$E_{d3} = \begin{bmatrix} 1 & 0 \\ -0.2015 & 1.3194 \end{bmatrix}.$$

VII. COMPUTER SIMULATIONS II: AFTER REDESIGN

The simulations of the redesigned digitally controlled Chua's circuit, described by (47)–(49), are shown in Figs. 15–19.

It can be seen that the digitally redesigned controller directs the trajectories away from the chaotic attractor and drives them to approach the desired unstable limit cycle. The performance of the redesigned digitally controlled Chua's circuit is much better than that of the original digitally controlled circuit—it was not working at all!.

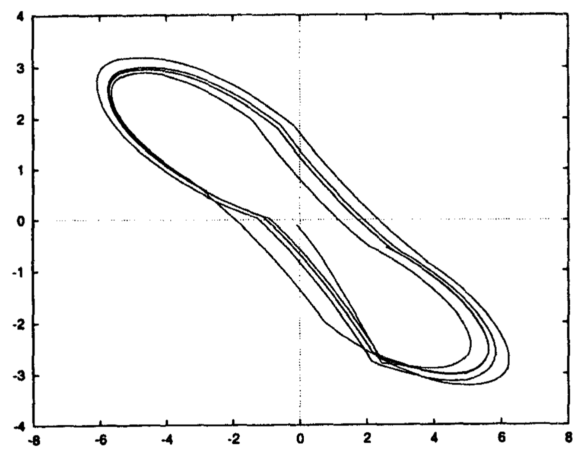

Fig. 18. Trajectory of Chua's circuit under digitally redesigned controller, projected onto x_3-x_1 plane, with sampling period $T = 0.4$ s.

VIII. CONCLUSIONS

An approach to controlling the chaotic Chua's circuit via digital redesign of an existing analog controller, using a relatively long sampling period, has been formulated, analyzed, and simulated in this paper. The digitally controlled Chua's circuit with different sampling periods have been compared. It has been shown that when the sampling period is relatively long, the trajectory of the digitally controlled Chua's circuit cannot approach the target orbit (an unstable limit cycle) of the original continuous-time controlled circuit and, as a result, the chaos control fails. To improve this critical digital control situation, the digital redesign technique proposed in this paper yields an equivalent digital controller for the same circuit, but can succeed the chaos control with a satisfactory performance. It is our hope that this digital redesign method can also be applied to other continuous-time nonlinear chaotic dynamical systems.

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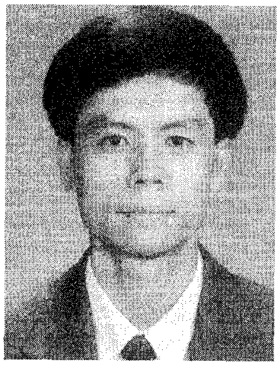

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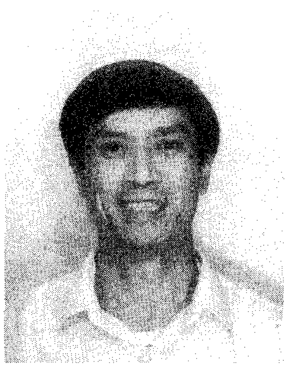

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