International Journal of Bifurcation and Chaos, Vol. 9, No. 4 (1999) 705–711

SYNCHRONIZING HIGH DIMENSIONAL CHAOTIC SYSTEMS VIA EIGENVALUE PLACEMENT WITH APPLICATION TO CELLULAR NEURAL NETWORKS

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Received October 5, 1997; Revised July 31, 1998

In this paper a method for synchronizing high dimensional chaotic systems is developed. The objective is to generate a linear error dynamics between the master and the slave systems, so that synchronization is achievable by exploiting the controllability property of linear systems. The suggested approach is applied to Cellular Neural Networks (CNNs), which can be considered as a tool for generating complex hyperchaotic behaviors. Numerical simulations are carried out for synchronizing CNNs constituted by Chua's circuits.

1. Introduction

Synchronization of chaotic systems has been the subject of many papers over the last few years [Carroll & Pecora, 1991; Cuomo *et al.*, 1993; Wu & Chua, 1994; Suykens & Vandewalle, 1997; Grassi & Mascolo, 1997]. However, most of the developed methods concern the synchronization of low dimensional systems, that is, dynamic systems with only one positive Lyapunov exponent. Since these systems are characterized by dynamics of limited complexity, the attention has been recently focused on higher dimensional chaotic systems. In fact, dynamic systems with several positive Lyapunov exponents exhibit more complex dynamics, which can be exploited for secure communications [Horio & Suyama, 1995; Arena *et al.*, 1996; Caponetto *et al.*, 1996; Milanovic & Zaghloul, 1996], pattern formation [Shalfeev & Kuznetsov, 1996] and active wave propagation [Chua *et al.*, 1995].

The aim of this paper is to make a contribution to the synchronization of high dimensional chaotic systems. The idea is to generate a linear time-invariant error system between the master and the slave systems. This objective is achieved by exploiting a suitable nonlinear synchronizing signal. In this way, synchronization can be achieved via eigenvalue placement if some structural properties related to the error system hold. The suggested approach is applied to Cellular Neural Networks (CNNs), which can be considered as a tool for generating hyperchaotic behaviors. Namely, by considering simple chaotic units as neural cells, the interconnections of a sufficiently large number of units can exhibit extremely complex behaviors, such as high-dimensional chaotic attractors [Chua *et al.*, 1995]. The paper is organized as follows. Some basic notions concerning synchronization are summarized in Sec. 2, whereas in Sec. 3 two high dimensional

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chaotic systems are coupled so that a linear time-invariant synchronization error system is generated. Therefore hyperchaos synchronization is achieved by checking the controllability property of linear systems. In Sec. 4 it is shown that the proposed technique can be utilized for synchronizing a large class of hyperchaotic CNNs. Finally, in Sec. 5 numerical simulations are carried out for hyperchaotic CNNs constituted by Chua's circuits.

2. Synchronization: Statement of the Problem

Definition 1. Given two chaotic systems

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}) \tag{1}$$

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) \tag{2}$$

they are said to be synchronized if

$$\mathbf{e}(t) = (\mathbf{y}(t) - \mathbf{x}(t)) \to \mathbf{0} \quad \text{as } t \to \infty \tag{3}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{y} \in \Re^n$, $\mathbf{g} : \Re^n \to \Re^n$ and $\mathbf{e}$ is the synchronization error [Wu & Chua, 1994].

In order to obtain synchronization, it is necessary to consider a *coupling* between the chaotic systems (1) and (2). In particular, system (2) has to receive a signal $\mathbf{s}(\mathbf{x})$ from system (1). This signal can be viewed as a properly designed output of system (1). More precisely, the following definition of *master–slave synchronization* is given [Suykens & Vandewalle, 1997].

Definition 2. System (1) with output $\mathbf{s} : \Re^n \to \Re^m$ and the dynamic system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) + \mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y}), \tag{4}$$

where $\mathbf{l} \in \Re^n$ is a suitable nonlinear coupling, are said to be synchronized in master–slave configuration if $\mathbf{y} \to \mathbf{x}$ as $t \to \infty$. This implies that the error system between the *master* (1) and the *slave* (4)

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{g}(\mathbf{y}) + \mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y}) - \mathbf{g}(\mathbf{x})\\ &= \mathbf{g}(\mathbf{x}+\mathbf{e}) + \mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{x}+\mathbf{e}) - \mathbf{g}(\mathbf{x})\\ &= \mathbf{h}(\mathbf{e}, t)\end{aligned} \tag{5}$$

has a (globally) asymptotically stable equilibrium point for $\mathbf{e} = \mathbf{0}$. Note that synchronization is said to be *global* if $\mathbf{y} \to \mathbf{x}$ for any initial condition $\mathbf{y}(0)$, $\mathbf{x}(0)$ [Grassi & Mascolo, 1997].

3. Synchronization of High Dimensional Chaotic Systems

Now the attention is focused on the synchronization of hyperchaotic dynamics generated by high dimensional systems. In particular, the systems considered herein are described by the following set of differential equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}(\mathbf{x}) + \mathbf{c} \tag{6}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A} \in \Re^{n\times n}$, $\mathbf{B} \in \Re^{n\times m}$, $\mathbf{c} \in \Re^n$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T \in \Re^m$ with $m \le n$. Taking into account the considerations reported in the previous section, it is clear that, given two hyperchaotic systems in master–slave configuration

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}(\mathbf{x}) + \mathbf{c} \tag{7}$$

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{f}(\mathbf{y}) + \mathbf{c} + \mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y}) \tag{8}$$

the key problem is the choice of a suitable coupling $\mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y})$ able to synchronize (7) and (8). To this purpose, the following theorem is proved.

Theorem. *Given systems (7) and (8), let*

$$\mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y}) = \mathbf{B}(\mathbf{s}(\mathbf{x}) - \mathbf{s}(\mathbf{y})) \tag{9}$$

be the nonlinear coupling, where

$$\mathbf{s}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{K}\mathbf{x} \tag{10}$$

is the synchronizing signal, with

$$\mathbf{x} = \begin{bmatrix} k_{1,1} & k_{1,2} & \ldots & k_{1,n}\\ k_{2,1} & k_{2,2} & \ldots & k_{2,n}\\ \ldots & \ldots & \ldots & \ldots\\ k_{m,1} & k_{m,2} & \cdots & k_{m,n}\end{bmatrix} \in \Re^{mxn}.$$

Then the error system (5) becomes linear time-invariant and can be written as

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{e} - \mathbf{B}\mathbf{K}\mathbf{e} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{u} \tag{11}$$

where $\mathbf{u} = -\mathbf{K}\mathbf{e} \in \Re^m$ *plays the role of a state feedback.*

Proof. By substituting Eqs. (7)–(10) in Eq. (5), the error system becomes

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{g}(\mathbf{y}) + \mathbf{l}(\mathbf{s}(\mathbf{x}), \mathbf{y}) - \mathbf{g}(\mathbf{x})\\ &= \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{f}(\mathbf{y}) + \mathbf{c} + \mathbf{B}(\mathbf{s}(\mathbf{x}) - \mathbf{s}(\mathbf{y}))\\ &\quad - (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}(\mathbf{x}) + \mathbf{c})\\ &= \mathbf{A}\mathbf{e} + \mathbf{B}(\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}))\\ &\quad + \mathbf{B}(\mathbf{f}(\mathbf{x}) + \mathbf{K}\mathbf{x} - \mathbf{f}(\mathbf{y}) - \mathbf{K}\mathbf{y})\\ &= \mathbf{A}\mathbf{e} - \mathbf{B}\mathbf{K}\mathbf{e} = \mathbf{A}\mathbf{e} + \mathbf{B}\mathbf{u}\end{aligned} \qquad \blacksquare$$

Now, some results from linear system theory can be exploited in order to synchronize systems (7) and (8) via a suitable choice of the matrix $\mathbf{K}$. In particular, it is well known that system (11) can be transformed to the Kalman controllable canonical form [Kailath, 1980; Brogan, 1991]:

$$\dot{\overline{\mathbf{e}}} = \begin{bmatrix} \overline{\mathbf{A}}_c & \overline{\mathbf{A}}_{12} \\ \mathbf{0} & \overline{\mathbf{A}}_{nc} \end{bmatrix} \overline{\mathbf{e}} + \begin{bmatrix} \overline{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u} \qquad (12)$$

where the eigenvalues of $\overline{\mathbf{A}}_c$ are controllable (i.e., *they can be placed anywhere* by state feedback $\mathbf{u} = -\mathbf{Ke}$), whereas the eigenvalues of $\overline{\mathbf{A}}_{nc}$ are uncontrollable (i.e., they are not affected by the introduction of any state feedback). Therefore a necessary and sufficient condition to globally asymptotically stabilize system (11) is that its uncontrollable eigenvalues, if any, lie in the left half plane [Kailath, 1980; Brogan, 1991]. Since $\overline{\mathbf{e}} \to \mathbf{0}$ implies $\mathbf{e} \to \mathbf{0}$, it follows that and $\mathbf{y} \to \mathbf{x}$, $\mathbf{s}(\mathbf{y}) \to \mathbf{s}(\mathbf{x})$ and $\mathbf{l} \to \mathbf{0}$ as $t \to \infty$, that is, global synchronization is achieved between the high dimensional systems (7) and (8).

Remark 1. If system (11) is controllable, then all the modes can be arbitrarily assigned. As a consequence, synchronization can be achieved according to any specified feature.

Remark 2. Differently from [Carroll & Pecora, 1991], the proposed method does not require the computation of any Lyapunov exponent in order to verify synchronization. Moreover, since global synchronization is achieved, the suggested technique does not require initial conditions belonging to the same basin of attraction.

Remark 3. The proposed synchronization scheme is robust with respect to a parameter variation Δf_i in the nonlinearity f_i $(i = 1, \ldots, m)$ of the hyperchaotic system. This property is closely related to the synchronizing signal considered herein. Namely, let $\Delta_1\mathbf{f}(\mathbf{x})$ be the parameter variation in $\mathbf{f}(\mathbf{x})$ and let $\Delta_2\mathbf{f}(\mathbf{y})$ be the parameter variation in $\mathbf{f}(\mathbf{y})$. Clearly, the parameter variation $\Delta_1\mathbf{f}(\mathbf{x})$ will affect the synchronizing signal, that is, $\mathbf{s}_1(\mathbf{x}) = (\mathbf{f}(\mathbf{x}) + \Delta_1\mathbf{f}(\mathbf{x})) + \mathbf{Kx}$. It follows that the coupling (9) becomes $\mathbf{B}(\mathbf{s}_1(\mathbf{x}) - \mathbf{s}_2(\mathbf{y}))$, with $\mathbf{s}_2(\mathbf{y}) = (\mathbf{f}(\mathbf{y}) + \Delta_2\mathbf{f}(\mathbf{y})) + \mathbf{Ky}$. The corresponding error system is:

$$\begin{aligned}\dot{\mathbf{e}} &= \mathbf{Ay} + \mathbf{B}(\mathbf{f}(\mathbf{y}) + \Delta_2\mathbf{f}(\mathbf{y})) + \mathbf{c} + \mathbf{B}(\mathbf{s}_1(\mathbf{x}) \\ &\quad - \mathbf{s}_2(\mathbf{y})) - (\mathbf{Ax} + \mathbf{B}(\mathbf{f}(\mathbf{x}) + \Delta_1\mathbf{f}(\mathbf{x})) + \mathbf{c}) \\ &= \mathbf{Ae} - \mathbf{BKe}\end{aligned}$$

that is, the method is intrinsically robust with respect to a parameter variation in the nonlinearity.

4. Synchronizing Hyperchaotic CNNs

In order to illustrate the proposed method, in this Section CNNs are considered as an example of high dimensional dynamic system. It is well known that a CNN is an array of simple circuits particularly suitable for VLSI implementation [Chua & Yang, 1988]. The interconnections of a sufficiently large number of simple units can exhibit extremely complex behaviors, such as Turing patterns, spiral and scroll waves or high-dimensional chaotic attractors [Chua *et al.*, 1995]. Since the attention is focused on hyperchaotic dynamics, the following definitions and assumptions are reported.

Definition 3. An N-cell CNN is defined mathematically by four specifications [Chua *et al.*, 1995]:

(1) CNN cell dynamics;
(2) CNN synaptic law;
(3) Initial conditions;
(4) Boundary conditions.

The basic CNN cell is shown in Fig. 1. It contains, in addition to the dynamical circuit core characterized by its state vector $\mathbf{x}_i$, an input $\mathbf{u}_i$, a dc bias $\mathbf{z}_i$, an output $f(\mathbf{x}_i)$ and a synaptic input current $\mathbf{I}_i$ describing the interactions among cells.

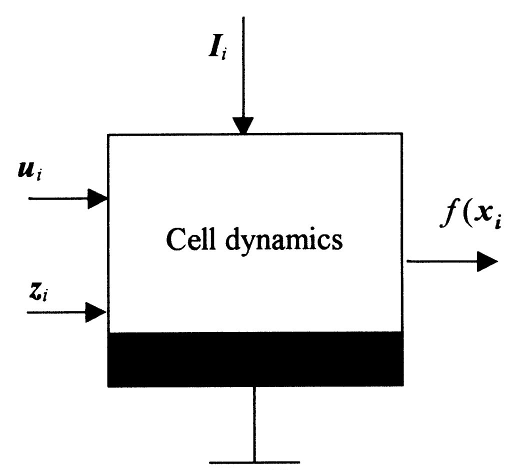

Fig. 1. The basic CNN cell.

4.1. *CNN cell dynamics*

Assumption 1. The ith cell is a dynamic circuit described by a set of M differential equations:

$$\dot{\mathbf{x}}_i = \mathbf{C}\mathbf{x}_i + \mathbf{d}f(\mathbf{x}_i) + \mathbf{z} \quad \text{for } i = 1, \ldots, N \qquad (13)$$

where $\mathbf{x}_i \in \Re^M$ is the state vector, $\mathbf{z} \in \Re^M$ is the dc bias, $\mathbf{C} \in \Re^{M\times M}$ is a constant matrix, $\mathbf{d} \in \Re^M$ is a constant vector and $f(\mathbf{x}_i)$ is the scalar output, with $f : \Re^M \to \Re$.

Remark 4. Several chaotic circuits can be modeled by (13). For instance, Chua's circuit [Chua *et al.*, 1995], the n-dimensional Chua's circuit [Gotz *et al.*, 1993], and the hyperchaotic circuits in [Matsumoto *et al.*, 1986; Tamasevicius *et al.*, 1996; Tamasevicius *et al.*, 1997] are examples of systems satisfying Assumption 1. Hence, they can be used as building blocks to obtain hyperchaotic CNNs.

4.2. *CNN synaptic law*

It is worth noting that the CNN synaptic law depends on the input and the state of all cells located within a prescribed sphere of influence, or *neighborhood size* [Chua *et al.*, 1995]. Although the contribution from the input and the state of each neighbor cell may be any arbitrary nonlinear coupling, in the following, linear interactions are considered.

Assumption 2. The synaptic law of the ith CNN cell is:

$$\mathbf{I}_i = \mathbf{H}_i[\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_N^T]^T \quad \text{for } i = 1, \ldots, N \quad (14)$$

where $\mathbf{I}_i \in \Re^M$ is the synaptic current whereas $\mathbf{H}_i \in \Re^{M\times MN}$ is a sparse matrix, which takes into account the *local connection* among cells.

Remark 5. Following the conjecture in [Kapitaniak *et al.*, 1994], it is convenient to choose the elements of the matrix $\mathbf{H}_i$ sufficiently small. In fact, given N chaotic subsystems, if the coupling is not too strong, the whole array is expected to be characterized by N positive Lyapunov exponents.

4.3. *Initial and boundary conditions*

Given the cell dynamics (13) and the synaptic law (14), it is not difficult to find a set of initial and boundary conditions able to generate hyperchaotic behavior. For instance, in [Kapitaniak *et al.*, 1994] hyperchaos is obtained by considering the initial conditions for which each cell is chaotic and by taking ring boundary conditions.

Now, it can be easily shown that systems described by (6) can include CNNs as a particular case. In fact, by considering (13) and (14), the whole CNN becomes:

$$\dot{\mathbf{x}}_i = \mathbf{C}\mathbf{x}_i + \mathbf{d}f(\mathbf{x}_i) + \mathbf{z} + \mathbf{I}_i \quad \text{for } i = 1, \ldots, N \qquad (15)$$

Moreover, by defining the following matrices and vectors:

$$\begin{gathered}
\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_N^T]^T \in \Re^{MN}, \\
\mathbf{H} = [\mathbf{H}_1^T, \mathbf{H}_2^T, \ldots, \mathbf{H}_N^T]^T \in \Re^{MN\times MN}, \\
\mathbf{A} = (\operatorname{diag}(\mathbf{C}) + \mathbf{H}) \in \Re^{MN\times MN}, \\
\mathbf{B} = \operatorname{diag}(\mathbf{d}) \in \Re^{MN\times N}, \\
\mathbf{c} = [\mathbf{z}^T, \mathbf{z}^T, \ldots, \mathbf{z}^T]^T \in \Re^{MN}, \\
\mathbf{f}(\mathbf{x}) = (f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_N))^T \in \Re^N,
\end{gathered}$$

it is clear that the CNN can be rewritten in the form (6), with $n = MN$ and $m = N$. As a consequence, the approach developed in Sec. 3 can be exploited for synchronizing hyperchaotic CNNs in the master–slave form (7)–(8).

5. Example

A 5-cell CNN consisting of identical Chua's circuits forming a ring is considered [Kapitaniak *et al.*, 1994]. The cell dynamics (13) is a set of 3 differential equations with

$$\mathbf{C} = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 2.95 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{z} = \mathbf{0},$$

$$f(\mathbf{x}_i) = f(x_{3i-2}) = |x_{3i-2} + 1| - |x_{3i-2} - 1|, \quad i = 1, \ldots, 5. \qquad (16)$$

By considering the synaptic law

$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \\ \cdots \\ \mathbf{I}_5 \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \cdots \\ \mathbf{x}_5 \end{bmatrix}$$

where

$$\mathbf{H} = \begin{bmatrix} -\hat{\mathbf{H}} & \hat{\mathbf{H}} & \mathbf{0} & \mathbf{0} & 0 \\ \mathbf{0} & -\hat{\mathbf{H}} & \hat{\mathbf{H}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\hat{\mathbf{H}} & \hat{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{H}} & \hat{\mathbf{H}} \\ \hat{\mathbf{H}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\hat{\mathbf{H}} \end{bmatrix}$$

and

$$\hat{\mathbf{H}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the master CNN (7) is obtained, for which experimental observation of hyperchaos have been reported in [Kapitaniak *et al.*, 1994]. Two projections of the hyperchaotic attractors are shown in Figs. 2 and 3, respectively. Moreover, by considering (9) and (10), the slave CNN (8) is derived.

Since the controllability matrix $\lfloor \mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B} \cdots \mathbf{A}^{14}\mathbf{B} \rfloor$ of (11) is full rank, the synchronization error system is controllable and its eigenvalues can be placed anywhere [Kailath, 1980; Brogan, 1991]. Therefore, it is possible to compute a feedback matrix $\mathbf{K} \in \Re^{5\times 15}$ such that $\mathbf{y} \to \mathbf{x}$ *as* $t \to \infty$ for *any initial state*. For instance, the set of the error system eigenvalues becomes $\{-1, -1, -1, -1, -1, -2, -2, -2, -2, -2, -3, -3, -3, -3, -3\}$ for

$$\mathbf{K} = \begin{bmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_1 & \mathbf{k}_2 \\ \mathbf{k}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_1 \end{bmatrix} \tag{17}$$

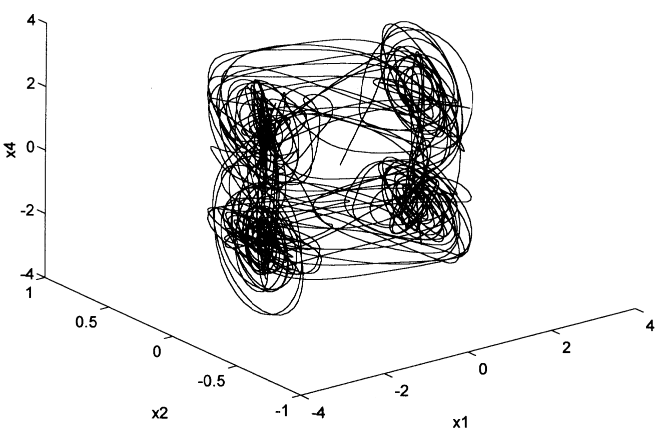

Fig. 2. Projection on (x_1, x_2, x_4) of the attractor generated by the CNN with $N = 5$.

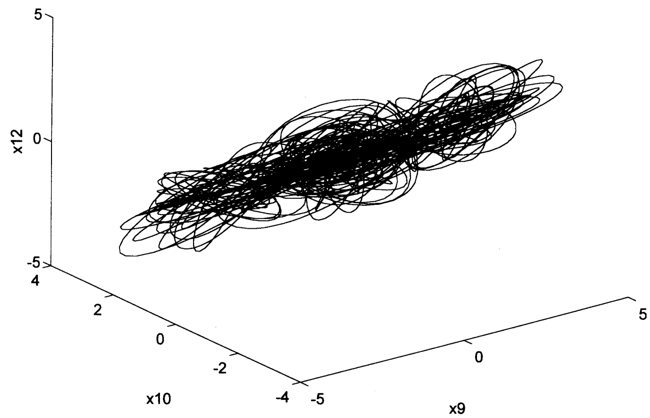

Fig. 3. Projection on (x_9, x_{10}, x_{12}) of the attractor generated by the CNN with $N = 5$.

where $\mathbf{k}_1 = [0.6068\ 0.3695\ 1.5547]$, $\mathbf{k}_2 = [0.0034\ 0.0135\ 0.0034]$, $\mathbf{0} \in \Re^{1\times 3}$. Figures 4 and 5 show the synchronization between the variables (x_4, y_4) and (x_{12}, y_{12}), respectively. Similar results have been obtained by considering CNNs with a larger number of cells. For example, some simulations have been carried out for a CNN with 100 cells. In particular, the synchronization between the variables (x_{88}, y_{88}) is reported in Fig. 6.

Remark 6. It can be interesting to discuss about the structure of the matrix $\mathbf{K}$. To this purpose, several simulations have been carried out for the CNN considered in the previous example. These computations have shown that, in the general case, $\mathbf{K}$ is a *full* rectangular matrix. However, in some cases, $\mathbf{K}$ can be characterized by a particular

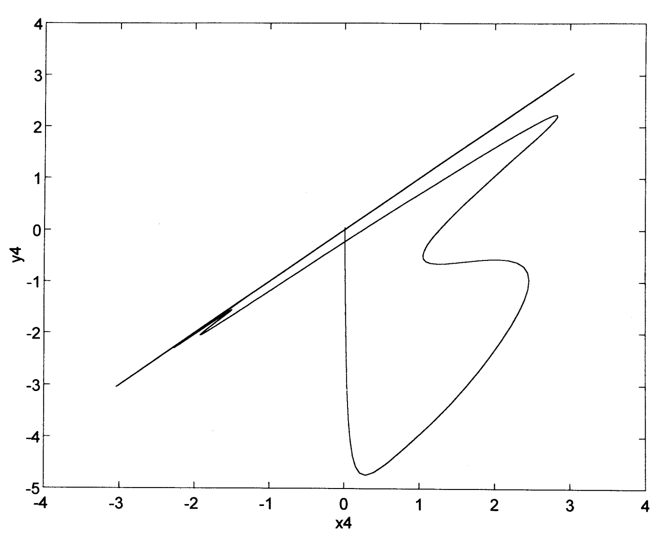

Fig. 4. Synchronization between the variables x_4 and y_4.

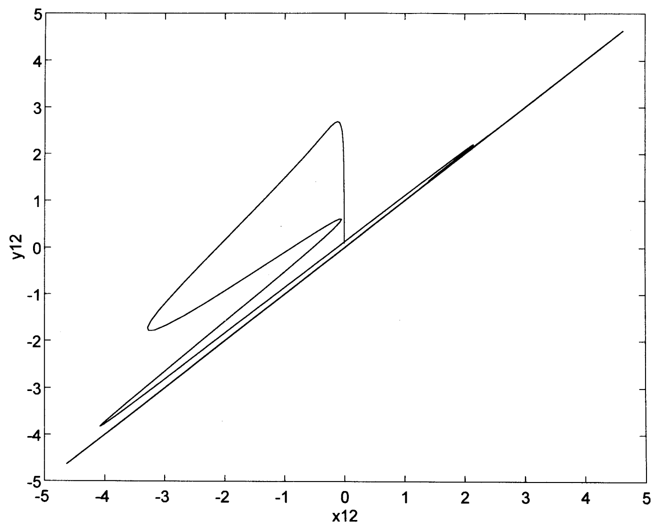

Fig. 5. Synchronization between the variables x_{12} and y_{12}.

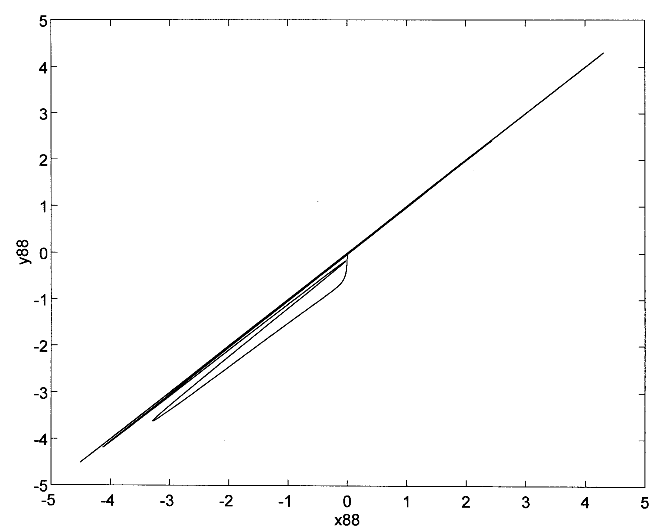

Fig. 6. Synchronization between the variables x_{88} and y_{88} for a CNN with $N = 100$.

structure, depending on the position of the eigenvalues in the complex plane as well as on the coupling between the CNN cells. For example, the matrix (17) is a block circulant matrix [Perfetti, 1997]. Unfortunately, numerical computations have failed in finding simpler structures for **K**, such as diagonal **K** matrices.

6. Conclusion

In this paper a method for synchronizing high dimensional chaotic systems has been developed and applied to CNNs constituted by chaotic cells. The proposed approach presents several interesting features. In particular: (1) It is *flexible*, because it enables synchronization to be achieved for different hyperchaotic behaviors, which can be obtained by changing the CNN cell dynamics or by adding new cells; (2) it does not require the computation of any Lyapunov exponent in order to verify synchronization; (3) it does not require CNN initial conditions belonging to the same basin of attraction. Simulation results for hyperchaotic CNNs constituted by Chua's circuits have confirmed the effectiveness of approach developed herein.

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