A Frequency Approach for Analyzing and Controlling Chaos in Nonlinear Circuits

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Abstract— **The paper presents a frequency domain approach for studying the chaotic dynamics of an important class of nonlinear circuits. By formulating an *elementary model of chaos* and using the *harmonic balance principle*, techniques for the analysis and the stabilization to a periodic solution of complex systems are developed. They result in *engineering tools* which are simple and practical, although not rigorous in principle, providing a qualitative view of the global dynamics in study. Some applications concerning the recent unfolded Chua's circuit are considered.**

I. Introduction

Over the past few decades, one of the most exciting and interesting ideas developed in nonlinear dynamics is that concerning the complex and chaotic behavior of systems. Chaos has been the object of intense research from all the scientific disciplines (biology, chemistry, mathematics, physics, engineering, etc.) as well as from theoreticians and natural philosophers. So, many advances have been made in capturing such complicate phenomena and in introducing differently oriented methods and techniques for the analysis of chaotic dynamics. To this regard, the availability of increasingly powerful computers played an essential role in describing the actual behaviors and in encouraging the developments of new approaches to the problem.

Recently, after that a general knowledge of the essence and main characteristics of chaos has diffused in the scientific communities, *practical implications* of these ideas have been perceived and widely accepted in various technological fields. This fact leads to novel applications where the chaos has to be, in some sense, controlled and two possible paradigms, the first one negative since chaos must be avoided and the second one positive when chaos is a desired dynamics, are considered in a growing number of contributions [1]. In particular, there is no surprise that challenging and promising new directions of research appear in engineering problems and in particular in the area of system and circuit theory and applications (see, for example [2], [3]).

In this framework, the purpose of this paper is to give a contribution to the treatment of system with chaotic dynamics presenting an *engineering approach* to the problem. It is a frequency domain approach which comes from classical methods familiar in electronic and mechanical fields, so being particularly appropriate to connect together new and conventional concepts of nonlinear dynamics.

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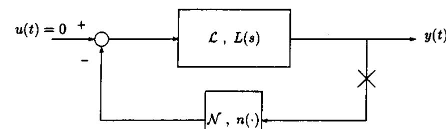

Fig. 1. Basic feedback system.

The presented approach is founded on the *harmonic balance principle* [4] (the same of the well-known describing function method), and using a *basic idea on an elementary chaos model* it leads to *practical techniques* for analysis and control. Such techniques have an engineering flavor, not being rigorous in nature, since they should allow one to reach qualitative results of reasonable accuracy on the global system dynamics at low computational cost. In particular, the prediction procedures have been widely applied to many classical and new chaotic systems with the *empirical evidence* of good reliability of the results [5], [6].

In this paper, the treatment has been limited to one class of dynamical systems of particular interest in the area of system and circuit theory. Sections II and III briefly present the analysis techniques, while the control problem, which takes into account some needs of periodic stabilization of chaos, is exposed with some detail in Section IV leading to a guaranteed design of a feedback controller. As a conclusion, in order to illustrate the application of the proposed approach for analysis and especially for controlling chaotic dynamics, some cases are considered and discussed in Section V with reference to the recent unfolded Chua's circuit which exhibits a very rich variety of complex behaviors [7].

II. System Representation and Main Characteristics

Consider *the basic feedback structure* of Fig. 1 where $\mathcal{L}$ is a linear time-invariant dynamic system and $\mathcal{N}$ is a nonlinear time-invariant static and memoryless system. The block $\mathcal{L}$ can be described by its transfer function

$$L(s) \doteq \frac{p(s)}{q(s)} \tag{1}$$

where s is the complex variable and $p(\cdot)$ and $q(\cdot)$ are polynomial operators, while the block $\mathcal{N}$ is represented by the nonlinear single-valued function $n(\cdot)$. The system of Fig. 1

is unforced and $y(t)$ denotes its scalar output. In terms of differential equation the system is governed by the form

$$q(D)y(t) + p(D)n[y(t)] = 0 \tag{2}$$

where D is the differential operator.

The feedback structure in study (see also [6]), well known in control engineering as Lur'e form [4], [8], is an important class of representation of circuits and systems also with regard to their chaotic dynamics.

Since the methods presented in the paper are based on a first harmonic study, assume for the system output the form

$$y_0(t) \doteq A + B\cos\omega t, \qquad B,\, \omega > 0 \tag{3}$$

and assume that the corresponding nonlinearity output (see Fig. 1) $n[y_0(t)]$ is expanded in Fourier series as

$$n[y_0(t)] = N_0(A,\, B)A + N_1(A,\, B)B\cos\omega t + \cdots. \tag{4}$$

The nonlinear system $\mathcal{N}$ is characterized, in an approximate form related to the (steady state) periodic regime, by the bias and ω frequency real gains

$$N_o(A,\, B) \doteq \frac{1}{2\pi A}\int_{-\pi}^{\pi} n[y_0(t)]\, d\omega\, t \tag{5}$$

$$N_1(A,\, B) \doteq \frac{1}{\pi B}\int_{-\pi}^{\pi} n[y_0(t)]\cos\omega t\, d\omega t \tag{6}$$

which are the well-known *describing function terms* [4], [8]. As an extension, one can define higher frequency complex gains $N_k(A,\, B)$, $k = 2, 3, \cdots$, which describe the remaining terms of (4).

Now, some definitions concerning constant and estimated periodic solutions of the system of Fig. 1 are introduced as main elements for the following paper developments.

- *Equilibrium Points* (EP's): the admissible constant output values $y = E_j$, $j = 1, 2, \cdots$ of the system. From (1) and (2) they solve the equation

$$y + n(y)L(0) = 0. \tag{7}$$

  The local stability features of the EP's E_j can be investigated by standard linear techniques substituting to the block $\mathcal{N}$ the gain (linearization)

$$n'(E_j) \doteq \left.\frac{dn(y)}{dy}\right|_{y=E_j}. \tag{8}$$

- *Predicted Limit Cycles* (PLC's): the approximate periodic solutions $y_0(t)$ of the system derived by the describing function method. According to (3), (5) and (6) these PLC conditions are

$$A[1 + N_0(A,\, B)L(0)] = 0 \tag{9}$$

$$1 + N_1(A,\, B)L(j\omega) = 0. \tag{10}$$

  Such equations follow by imposing *the harmonic balance principle* [4], [8] along the system loop of Fig. 1, where the transfer function L and the nonlinearity n have been evaluated at their steady state gains of zero and ω frequency. Equations (9) and (10) must be solved with respect to the parameters A, B and ω and, as a particular powerful interpretation, condition (10) graphically corresponds to the intersection of the Nyquist plot of $L(j\omega)$ with the function $-1/N_1$. In general, when B tends to zero the relation (10) expresses the Hopf bifurcation existence [9] and the relation (9) leads to a bias value $A = E_j$, being E_j the EP where the bifurcation occurs. Such a point can be viewed as *the generator of* a family of periodic solutions.

  The stability features concerning $y_0(t)$ can be evaluated from (9) and (10) by means of the Loeb criterion or similar procedures [10].

  Any periodic solution $y_0(t)$ indicates in the state space a limit cycle which is called *predicted* since it derives from a heuristic analysis and its exact shape and even existence are uncertain. The reliability of the prediction depends on the distortion along the system loop (see below).

- *Distortion*: the amount of the neglected higher harmonics concerning a PLC $y_0(t)$ of frequency ω. It can be expressed by the quantity

$$\Delta \doteq \frac{\|\tilde{y}_0(t) - y_0(t)\|_2}{\|y_0(t)\|_2}. \tag{11}$$

  Here, the symbol $\|\cdot\|_2$ denotes the L_2 norm on the period $2\pi/\omega$ and $\tilde{y}_0(t)$ is the steady-state periodic output of the system that is obtained when the closed loop is broken just before $\mathcal{N}$ (see Fig. 1) and the signal $y_0(t)$ is injected into $\mathcal{N}$.

  Small values of Δ, in a symbolic way $\Delta < \eta$, indicate that the open loop system is an efficient *low-pass filter* and that the corresponding PLC is reliable. Indeed, rigorous arguments can be used to guarantee the existence of a true periodic solution in a defined neighborhood of the predicted one under suitable conditions on Δ [4], [8].

The introduced elements are generally easy to compute even for high-order systems (distributed parameters or experimental data), due to the use of input-output models in Fig. 1, and in simple cases, they result in analytical forms.

III. The Prediction of Chaotic Dynamics

The use of the elements considered in Section II to detect conditions of complex dynamics is based on the essential idea that, roughly speaking, a chaotic behavior is a kind of *noisy limit cycle* [11]. This means that there exists some *mechanism of perturbation* which operates on conditions where periodic motions tend to occur.

By considering the several types of transition that take place from regular to chaotic regimes when a suitable parameter is moved, two phenomena appear to be important in relation to the point of interest, and they have the following geometric characterization:

1) *the homoclinic orbit*, which is a trajectory starting from an unstable equilibrium point along an outgoing eigenvector and returning to it along an incoming eigenvector [9]

2) *the period doubling*, which appears when a stable periodic trajectory shows a double loop due to the emergence of a component of twice period [9].

Both of these situations can be viewed as cases where a possible stable limit cycle respectively interferes 1) with an equilibrium point close to it of unstable characteristics and 2) a stable periodic perturbation of twice period.

In this framework, the objective is now that of deriving mathematical relations, approximate in nature, which correspond to the situations 1) and 2) intended as *leading indicators* of a chaotic behavior of a system [11], [5], [6].

According to the considerations of Section II the common existence of a potential limit cycle can correspond to the existence of a PLC with a distortion not sufficiently small, that is $\Delta > \eta$. Otherwise, for $\Delta < \eta$ the PLC indicates a true limit cycle. Therefore, the proposed conditions for two independent and alternative mechanisms of chaos onset are written as in the following:

1) *interaction PLC-EP (homoclinic orbit)*:

limit cycle
- there exists a PLC $y_0(t)$ satisfying (9) and (10),
- which is stable (Loeb or similar criterion),
- and such that $\Delta > \eta$,

perturbation
- there exists an EP E_j satisfying (7),
- which is unstable (linearization),
- and such that:
$$E_j = A + B\cos\omega t = y_0(t), \text{ for some } t. \tag{12}$$

The last condition expresses the fact that the EP, which must be different from the generator of the PLC, falls within the projection onto the y axis of the same PLC $y_0(t)$. This is introduced as a *closeness condition* since otherwise the presence of the EP tends to be not important for the PLC dynamics [11], [5], [6].

2) *period doubling*:

limit cycle
- there exists a PLC $y_0(t)$ satisfying (9) and (10),
- which is stable (Loeb or similar criterion),
- and such that $\Delta > \eta$,

perturbation
- it is:
$$1 + N_{\frac{1}{2}}(A, B, \vartheta)L(j\omega/2) = 0. \tag{13}$$

The last condition (13) results from the harmonic balance of $\omega/2$ frequency terms along the loop of Fig. 1. It follows from the assumption that a small $\omega/2$ component is added to the PLC so that the nonlinear system $\mathcal{N}$ can be described by its *complex incremental gain*, namely $N_{\frac{1}{2}}$, as

$$N_{\frac{1}{2}}(A, B, \vartheta) \doteq \frac{1}{\pi}\int_{-\pi}^{\pi} n'[y_0(t)]\cos(\frac{\omega}{2}t - \vartheta) e^{j(\omega/2t-\vartheta)}\, d\frac{\omega}{2}t. \tag{14}$$

This relation is similar to (5) and (6) defining N_o and N_1: the presence of the derivative n' is due to the linearization of n around $y_0(t)$ and ϑ denotes the delay phase of the $\omega/2$ perturbation with respect to the PLC. In graphical terms the solution of (13) corresponds to the intersection of the Nyquist plot of $L(j\omega)$ with a suitable circle defined by $N_{\frac{1}{2}}$ occurring at a frequency which is the middle of that defining the PLC [10], [5], [6].

The two sets of conditions concerning the simple models 1) and 2) can be solved using the main system characteristics of Section II, that is PLC's and EP's, in the basic equations (12) and (13). So, it is possible to find the parameter regions where the above complex phenomena are estimated and where chaotic behaviors can be expected. In addition, the state space location of these motions is derived and in simple cases all the results are obtained in analytical form [11], [5], [6].

In such a global view of the possible dynamics exhibited by a class of systems, the distortion Δ takes a quantitative meaning since it can be thought as a *free parameter* to be varied and for which the limit η of the transition between periodic ($\Delta < \eta$) and chaotic ($\Delta > \eta$) solutions results precisely defined.

The prediction techniques described in this section represent for the chaos what the describing function method is for the limit cycles. They have been largely applied to many classical and new chaotic systems by giving in any case qualitatively correct results, of reasonable accuracy, with quite simple computations. More complicate chaotic attractors have been derived by *componing the elementary models* described by situations 1) and 2) [11], [5], [6].

IV. The Control of Chaotic Dynamics

In the general topic of controlling chaos one of the most studied problem is the system *stabilization to a periodic solution* (see, for example [2] and [12]). In particular, the problem appears to be meaningful when some main characteristics must be preserved and hence the original system chaoticity introduces peculiar aspects of control.

Before presenting a precise statement it is important to remark that the formulation of the control problem, as well as its solution procedure, is strongly connected with the main idea introduced in the analysis of Section III, that is looking at the chaos as a noisy limit cycle. In a certain sense this idea will be used in an *inverse way* with respect to what is made in the analysis approach.

4.1. Problem Statement

Consider a system of the form shown in Fig. 1 and assume that it exhibits a chaotic behavior and a variety of other regular dynamics in the different regions of its state space. The aim is to design a control of this system such that the following specifications are satisfied:

a) *Stabilize the chaotic motions to an admissible periodic solution* $y_d(t)$. This means that the desired solution $y_d(t)$ must be connected in some way to the original behavior of the system. Typically, $y_d(t)$ can be fixed as an average of such chaotic motions, so tending to the limit cycle which is the basis of their model of Section III (noisy limit cycle). More generally one can also select a reduced amplitude $y_d(t)$ and force the system (in its parameter space) toward the Hopf bifurcation.

b) *Preserve the EP's* of the original system.

c) *Preserve the stability features* of the interesting EP's. This can correspond to maintain the same number of unstable eigenvalues for the chaotic and the controlled system or even to have the same eigenvalues.

Therefore, the control aim is to remove chaos (point a) without modifying the essential features of the given system (points b and c) far from this behavior. The chaos replacement is made in terms of the most similar regular regime, the periodic one. From a geometrical point of view the strange attractor [9], which can be thought as a *bundle* filled of trajectories, is substituted by a limit cycle. In the typical case the limit cycle is the average trajectory of the bundle and so the controlled process intuitively results in the nonchaotic system which is the closest one to the given (chaotic) system. This physically means that the control energy is small and easy transitions between the two operating modes are possible, as required in certain applications [1]. When reduced limit cycles are the goal of the control the strange attractor is pushed towards its nucleus and the system is moved away from the chaotic behavior. This can be required to ensure a regular dynamics for a larger range of certain parameters of the original system (robustness) [2].

4.2. Solution Structure

In order to solve the posed control problem a *feedback* compensator is selected as shown in Fig. 2. It is made by a nonlinear static block described by the *polynomial function*

$$n_c(y) = \sum_{i=1}^{h} \gamma_i y^i \qquad (15)$$

and by a *linear (washout) filter* [2] of transfer function

$$L_c(s) = \frac{s}{s+\lambda}, \qquad \lambda > 0. \qquad (16)$$

In (15) and (16) h is a suitable integer to be selected, while γ_i, $i = 1, 2, \cdots, h$ and λ are the unknown coefficients which define the control. This structure is the simplest one to produce a nonlinear action while having no steady state effect in constant regime [12], [13]. As made for n by formulas (5) and (6), the describing function real terms $N_{0c}(\gamma_i)$ and $N_{1c}(\gamma_i)$ of the nonlinearity n_c can be defined as well as the higher order complex harmonic gains $N_{kc}(\gamma_i)$, $k = 2, 3, \cdots$.

To satisfy the above control specifications a, b, and c, the following considerations and conditions are given concerning the system of Fig. 2:

a) Assume that the periodic solution to be attained has the form

$$y_d(t) \doteq A + B_d \cos \omega t, \qquad B_d, \omega > 0 \qquad (17)$$

where B_d is the desired amplitude, which can be fixed in the range between the average of the chaotic motions and a small value, and the parameters A and ω are not given a priori.

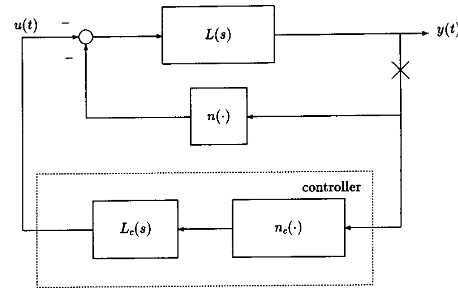

Fig. 2. Structure of the controlled chaotic system.

In order to have the periodic solution (17), impose now that its parameters satisfy the describing function method. In other words, according to Section II *assume that $y_d(t)$ is a PLC* for the system of Fig. 2. The related harmonic balance conditions simply result in

$$A[1+N_0(A, B_d)L(0)+N_{0c}(A, B_d, \gamma_i)L_c(0, \lambda)L(0)] = 0 \qquad (18)$$

$$1+N_1(A, B_d)L(j\omega)+N_{1c}(A, B_d, \gamma_i)L_c(j\omega, \lambda)L(j\omega) = 0. \qquad (19)$$

By considering that $L_c(0) = 0$ as shown by (16), the relations (18) and (19) can be rewritten as

$$A[1 + N_0(A, B_d)L(0)] = 0 \qquad (20)$$

$$N_{1c}(A, B_d, \gamma_i)L_c(j\omega, \lambda) = -[1 + N_1(A, B_d)L(j\omega)]/L(j\omega). \qquad (21)$$

The reliability of this PLC depends by the distortion (see Section II) Δ_c of the controlled system which is defined as in (11) by the relation

$$\Delta_c(\gamma_i, \lambda) \doteq \frac{\|\tilde{y}_d(t) - y_d(t)\|_2}{\|y_d(t)\|_2}. \qquad (22)$$

Now, $\tilde{y}_d(t)$ is the steady state system output when the loop is broken as shown in Fig. 2 and the input to the nonlinearities is $y_d(t)$.

Therefore, to guarantee that the solution (17) is *actually attained*, the controller is defined by those parameters corresponding to the condition

$$\min_{\gamma_1, \lambda} \Delta_c(\gamma_i, \lambda). \qquad (23)$$

b) The presence of the washout filter (16), with $L_c(0) = 0$, *structurally* ensures that the controlled system has the same EP's of the original system.

c) Since the washout filter (16) introduces one additional stable dynamic mode, a linearized analysis of the system in Fig. 2 shows that the conditions to maintain the stability features of any EP E_j are expressed in terms of inequalities on the slope of the controller nonlinearity as

$$-\frac{1}{\sigma_M(E_j, \lambda)} < n_c'(E_j, \gamma_i) < \frac{1}{\sigma_m(E_j, \lambda)}, \qquad j = 1, 2, \cdots, l \qquad (24)$$

In these relations, where l is the number of the EP's of control interest, the positive quantities σ_m and σ_M respectively denote the minimum and the maximum intersection with the real axis of the Nyquist plot of

$$\frac{L_c(j\omega,\, \lambda)L(j\omega)}{1+n'(E_j)L(j\omega)} \tag{25}$$

The conditions (24) can be strengthened by considering some margins of stability, until to impose $n'_c = 0$ that corresponds to maintain in the EP's all the same original eigenvalues.

Then, summarizing these considerations, the control specifications are essentially fulfilled when the compensator parameters are selected according to conditions (20), (21), (23), and (24). The sense of the proposed solution is that of i) synthesizing a feedback control in such a way that the describing function method just predicts the desired periodic solution ((20) and (21)) and ii) using the degrees of freedom of the controller to have a corresponding small distortion ((23)) and a negligible effect on the local stability features ((24)).

The typical case where B_d is fixed as the average of the chaotic motions has a particular meaning [12], [13]. Here, the desired solution $y_d(t)$ given by (17) is intuitively the PLC recognized as chaos by the analysis of Section III. Then, according to (10) computed for $B = B_d$, the right-hand side of (21) vanishes and the condition reduces to

$$N_{1c}(A,\, B_d,\, \gamma_i) = 0. \tag{26}$$

Therefore, the controller stops the ω frequency as well as the zero frequency terms (since $L_c(0) = 0$) maintaining the same original PLC's, and it makes small the system distortion by the introduction of suitable higher harmonics to reduce the ones existing in the given process. This effect leads to a true limit cycle, and it can be said that the role of the control is to remove the disturbing actions from the noisy limit cycle (chaotic regime), hence restoring the basic limit cycle (periodic regime). Reversely, in the analysis of Section III the perturbations able to transform a possible limit cycle (a PLC) in a chaotic behavior were investigated. In practice, B_d can be derived by prediction techniques as those of Section III or by experimental way. The number h and more generally the specific degree of the coefficients γ_i are selected taking into account the main harmonics of the nonlinearity n which have to be compensated.

Finally, it is important to remark, according to the related considerations of Section II, that the *rigorous existence* of the desired periodic solution (17) can be shown, also giving its *accuracy*, by an appropriate reduction of the corresponding system distortion [4], [8]. In this sense, while the analysis of Section III results in approximate conclusions, the proposed control solution, which is known as distortion control in [13], can be precisely justified.

4.3. Solution Procedure and Validation

With respect to the signal $y_d(t)$ given by (17) define the real h-vectors

$$\gamma \doteq [\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_h]^T \tag{27}$$

$$p(A,\, B_d) \doteq \frac{1}{\pi B_d}\int_{-\pi}^{\pi} [y_d(t) \quad y_d^2(t) \quad \cdots \quad y_d^h(t)]^T \cos\omega t\, d\omega t \tag{28}$$

where the symbol γ^T denotes the transpose of γ.
The describing function of n_c becomes (see (6) and (15))

$$N_{1c}(A,\, B_d,\, \gamma_i) = p^T\gamma. \tag{29}$$

Then, define the complex h-vectors

$$q_k(A,\, B_d) \doteq \frac{1}{\pi B_d}\int_{-\pi}^{\pi} [y_d(t) \quad y_d^2(t) \quad \cdots \quad y_d^h(t)]^T e^{jk\omega t}\, d\omega t, \quad k = 2,\, 3, \cdots \tag{30}$$

and recalling that N_k indicates the higher harmonics complex gains of the function n, introduce the $h \times h$ matrix D, the h-vector d and the scalar δ in the form

$$D(A,\, B_d,\, \omega,\, \lambda) \doteq \left(\frac{B_d^2}{2A^2 + B_d^2}\right) \cdot \sum_{k}^{2,3,\cdots} |L(jk\omega)L_c(jk\omega,\, \lambda)|^2 [q_k^*(A,\, B_d) q_k^T(A,\, B_d)] \tag{31}$$

$$d(A,\, B_d,\, \omega,\, \lambda) \doteq \left(\frac{B_d^2}{2A^2 + B_d^2}\right) \cdot \sum_{k}^{2,3,\cdots} |L(jk\omega)|^2\, Re\,[N_k^*(A,\, B_d)L_c(jk\omega,\, \lambda)q_k^T(A,\, B_d)] \tag{32}$$

$$\delta(A,\, B_d,\, \omega) \doteq \left(\frac{B_d^2}{2A^2 + B_d^2}\right) \sum_{k}^{2,3,\cdots} |L(jk\omega)|^2 |N_k(A,\, B_d)|^2 \tag{33}$$

where q_k^* denotes the conjugate of q_k.
It can be easily shown that the square of the distortion (22) can be written as

$$\Delta_c^2 = (A,\, B_d,\, \omega,\, \lambda) = \gamma^T D\gamma + 2d^T\gamma + \delta. \tag{34}$$

Finally, define the real function

$$\Phi(A,\, B_d) \doteq A[1 + N_0(A,\, B_d)L(0)] \tag{35}$$

the complex function

$$\Psi(A,\, B_d,\, \omega,\, \lambda) \doteq -\frac{[1 + N_1(A,\, B_d)L(j\omega)]}{L(j\omega)L_c(j\omega,\, \lambda)} \tag{36}$$

and with respect to the EP's E_j the real h-vectors

$$g(E_j) \doteq [1 \quad 2E_j \quad 3E_j^2 \quad \cdots \quad nE_j^{n-1}]^T, \quad j = 1,\, 2, \cdots, l. \tag{37}$$

Now, in terms of relations (29) and (34)÷(37) the complete control conditions (20), (21), (23) and (24) derived in Section 4.2 can be restated in the following optimization problem on the coefficients γ and λ

$$\min_{\gamma,\,\lambda} [\gamma^T D(A,\, B_d,\, \omega,\, \lambda)\gamma + 2d^T(A,\, B_d,\, \omega,\, \lambda)\gamma + \delta(A,\, B_d,\, \omega)] \tag{38}$$

subject to the constraints

$$p^T(A,\, B_d)\gamma = \Psi(A,\, B_d,\, \omega,\, \lambda) \tag{39}$$

$$\Phi(A,\, B_d) = 0 \tag{40}$$

$$-\frac{1}{\sigma_M(E_j,\, \lambda)} < g^T(E_j)\gamma < \frac{1}{\sigma_m(E_j,\, \lambda)}, \quad j = 1,\, 2, \cdots, l. \tag{41}$$

The important fact is that the linear dependence on γ of the controller has produced, for any fixed λ, a *standard quadratic optimization problem.* Indeed, the essential control action is just due to γ which must suitably shape the different higher harmonics to compensate those produced by the original nonlinearity n (Fig. 2). The role of λ is different and less significant since it only must limit the high frequency gain of the filter L_c of (16) without introducing undesirable dynamics in the solution.

Therefore, the solution procedure of the problem (38)$\div$(41) *separates the selection* of γ and λ according to the following steps:

1) From (40) derive

$$A = A(B_d) \doteq A_d \tag{42}$$

neglecting the case $A = 0$ since those of Section III are essentially asymmetric models of chaos.

2) Substitute (42) in (39) and consider the imaginary part of the obtained equation which is

$$Im\, \Psi(A,\, B_d,\, \omega,\, \lambda) = 0. \tag{43}$$

In the range of validity of (43) assume a value $\lambda = \lambda_d > 0$ which is sufficiently large to avoid the introduction of slow dynamics in the controlled system. Hence, from (43) determine the corresponding $\omega = \omega(B_d,\, \lambda_d) \doteq \omega_d$

3) Use A_d, ω_d, λ_d in (38), (39) and (41) leading to the problem

$$\min_{\gamma} \left[\gamma^T D(\Lambda_d, B_d, \omega_d, \lambda_d) + 2d^{\tau}(A_d, B_d, \omega_d, \lambda_d)\gamma\right] \tag{44}$$

subject to the constraints

$$p^T(A_d,\, B_d)\gamma = Re\, \Psi(A_d,\, B_d,\, \omega_d,\, \lambda_d) \tag{45}$$

$$-\frac{1}{\sigma_M(E_j,\, \lambda_d)} < g^T(E_j)\gamma < \frac{1}{\sigma_m(E_j,\, \lambda_d)}, \quad j = 1,\, 2, \cdots, l. \tag{46}$$

4) Solve the standard problem (44)$\div$(46) with respect to γ obtaining the controller parameters $\gamma = \gamma_d$ as a function of the given amplitude B_d. In the reasonable assumption that the number l of EP's of interest is not greater than the number h of the components of γ, the admissible set of this vector is shown to be *non empty*. In particular, since the first component γ_1 does not appear in (44) the problem is equivalent to a quadratic minimization with l linear inequality constraints on $h - 1$ variables.

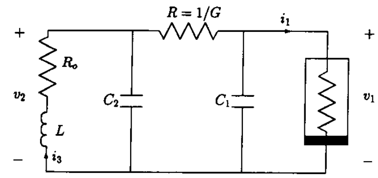

Fig. 3. The unfolded Chua's circuit.

5) Guarantee that the derived control system really attains the periodic solution $y_d(t)$ by verifying (i) the stability and (ii) the obtained distortion.
The point (i) can be investigated by one of the usual indicative stability criteria associated with the describing function method [10]. Certainly, it is important that the EP generating the PLC $y_d(t)$ is constrained to a suitable instability by one of the relations (46). Concerning point (ii) consider that the minimum distortion decreases with the number h of the control parameters. So, if it is possible, the value which ensures the existence of a true periodic solution close to the desired one $y_d(t)$ [4], [8] can be obtained by suitably increasing h.

V. Application to the Unfolded Chua's Circuit

Consider the Unfolded Chua's circuit drawn in Fig. 3 [7]. This is a recently presented electrical system, simply derived by the well-known Chua's circuit [14], which can exhibit the qualitative dynamics of any autonomous third-order system containing one odd symmetric three-segment piecewise-linear function. Its state equations result in

$$\begin{cases} \frac{dv_1}{dt} = -\frac{G}{C_1}v_1 + \frac{G}{C_1}v_2 - \frac{1}{C_1}\overline{n}(v_1) \\ \frac{dv_2}{dt} = \frac{G}{C_2}v_1 - \frac{G}{C_2}v_2 + \frac{1}{C_2}i_3 \\ \frac{di_3}{dt} = -\frac{1}{L}v_2 - \frac{R_0}{L}i_3 \end{cases} \tag{47}$$

where the nonlinear characteristic is written in the form

$$\overline{n}(y) = G_b y + \frac{1}{2}(G_a - G_b)(|y + E| - |y - E|). \tag{48}$$

The unfolded Chua's circuit can be represented in the basic feedback structure of Fig. 1. After a scaling transformation which defines the new time variable $(G/C_2)t$, assuming $y = v_1$ and

$$\alpha \doteq \frac{C_2}{C_1}, \quad \beta \doteq \frac{C_2}{LG^2}, \quad \rho \doteq \frac{C_2}{LG}R_0 \tag{49}$$

the transfer function of the linear part (see (1)) is found as

$$L(s) = \frac{\alpha[s^2 + (1+\rho)s + (\beta+\rho)]}{s^3 + (1+\alpha+\rho)s^2 + (\beta+\rho+\alpha\rho)s + \alpha\beta} \tag{50}$$

while the system nonlinearity results in $n(y) = \overline{n}(y)/G$.

Now, the purpose of the section is to present and discuss the application of the analysis and control techniques of Sections III and IV to some complex behaviors of this circuit.

TABLE I
COMPARISON OF PREDICTION RESULTS WITH ACTUAL BEHAVORS OF THE SYSTEM, WHERE A = APERIODIC (STABLE) MOTIONS, P = PERIODIC MOTIONS, PD = PERIOD DOUBLING, SS = SINGLE SCROLL CHAOS, DS = DOUBLE SCROLL CHAOS, U = UNSTABLE MOTIONS

ρ	stable PLC		interaction with $E_2 = 0$, i.e., $B \geq A$	predicted period doubling	distortion		system behavior
	A	B			Δ	Δ_2	
0	—	—	NO	—	—	—	A
−0.01	1.63	Hopf	NO	—	—	—	P
−0.10	1.44	0.95	NO	—	0.023	0.008	P
−0.15	1.34	1.00	NO	YES	0.029	0.009	P
−0.20	1.25	1.03	NO	—	0.034	0.008	P
−0.25	1.17	1.06	NO	—	0.038	0.007	PD
−0.30	1.10	1.07	NO	—	0.041	0.007	PD
−0.35	1.03	1.08	YES	—	0.044	0.007	SS
−0.40	0.96	1.10	YES	—	0.046	0.007	DS
−0.50	0.84	1.105	YES	—	0.049	0.009	DS
−0.60	0.73	1.108	YES	—	0.050	0.012	DS
−0.65	unstable	PLC	YES	—	—	—	U

5.1. Chaos Prediction

Assume in (50) and (48) the numerical values

$$\alpha = 6.7, \quad \beta = 14.3, \quad G_a/G = -1.2, \quad G_b/G = -0.7 \tag{51}$$

with the normalization $E = 1$ and regard as variable the parameter ρ which is moved on the negative axis. Observe that just the parameters α and β define the classical behavior of the original Chua's Circuit when $\rho = 0$ [14]. Recalling the characterization of Section III determine the EP's by solving the equation (see (7))

$$y + \left(1 + \frac{\rho}{\beta}\right) n(y) = 0. \tag{52}$$

For the parameter values (51) they result in

$$E_1 = -\left(\frac{7.15 + 0.5\rho}{4.29 - 0.7\rho}\right), \quad E_2 = 0, \quad E_3 = +\left(\frac{7.15 + 0.5\rho}{4.29 - 0.7\rho}\right) \tag{53}$$

and the characteristic equations concerning their local eigenvalues directly come from (50) and (48).

In the range of interest the transfer function $L(j\omega)$ has two intersections with the real axis at frequencies

$$[10.45 - 0.5\rho^2 \pm (0.52 - 28.6\rho - 18.15\rho^2 + 0.25\rho^4)^{1/2}]^{1/2} \tag{54}$$

corresponding (see (10)) to the oscillation frequencies of possible periodic solutions (the PLC's) of the system.

By solving the equations (9) and (10) for negative ρ it follows that the EP's E_1 and E_3 can generate two stable PLC's (symmetrically located) which are the only ones existing in the system.

According to Section III, these PLC's make evident candidate situations to chaos onset, in presence of perturbing mechanisms as 1) the interaction with an EP (homoclinic orbit) or 2) the twice period emergence.

To this regard the following Table I presents the obtained analysis results in terms of the main elements (only for the positive bias), by indicating when the above mechanisms are predicted and what is the corresponding actual behavior of the system. Observe that the EP at $E_2 = 0$ always results to be unstable, while the quantity Δ_2 expresses the distortion of the system without considering the contribution of the 2nd harmonic. The frequency of the PLC's corresponds to the negative sign in (54); it approximately varies in the interval (2.6, 3.0).

As a comment to Table I it can be said that the two considered indicators give good predictions of chaotic behavior with different characteristics. The homoclinic model (interaction PLC-EP) holds in an interval on ρ, that is approximately (−0.35, −0.60), where Δ is not sufficiently small and the estimate is correct. On the contrary, the period doubling model gives a specific value of ρ, that is $\rho \simeq -0.15$, which is considered to be at the beginning of the chaotic regime. Indeed, the true period doubling cascade arise when $-0.25 < \rho < -0.20$ but also this result appears a quite acceptable prediction. Moreover, from the data of Table I it is also possible to locate the chaos in the sense of sketching in state space the average of the strange attractors and considering its time oscillatory motion.

The above results are obtained from simple numerical computations: a straightforward fully analytical treatment could be developed in case of polynomial (cubic) representation of the nonlinearity [13].

Finally, Fig. 4(a) and 4(b) show two trajectory simulations of the systems respectively corresponding to values $\rho = -0.50$ and $\rho = -0.60$.

5.2. Chaos Control 1

Consider the circuit studied in Section 5.1 for $\rho = -0.50$. It clearly is a chaotic system (see Fig. 4(a)) with EP's at $E_2 = 0$ (one unstable eigenvalue) and at $E_1 = -1.49$, $E_3 = 1.49$ (two

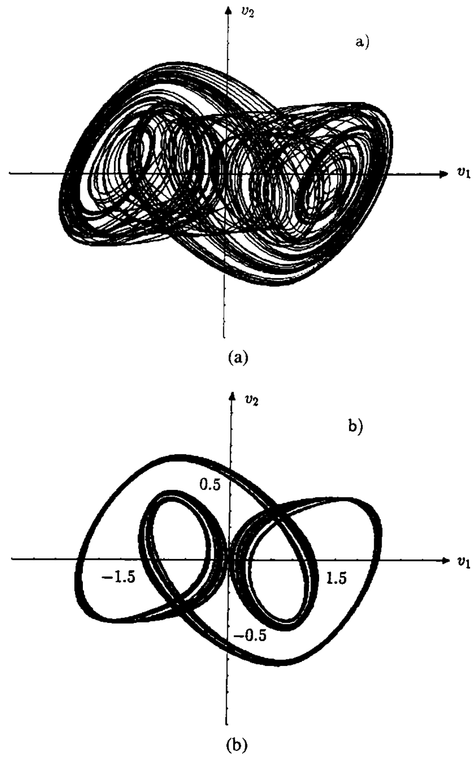

Fig. 4. Trajectory simulations of the unfolded Chua's circuit: (a) $\rho = -0.50$; (b) $\rho = -0.60$.

unstable eigenvalues), and two symmetrically located stable PLCs having parameters written in Table I.

Assume that this circuit must be controlled in such a way that the following occurs:

a) It attains a large periodic solution which is the average of the chaotic motions with positive bias. From the previous analysis (Table I) this corresponds to

$$y_d(t) = 0.84 + 1.105 \sin 2.67t \tag{55}$$

which is the PLC giving rise to chaos.

b) It has the EP's E_1, E_2, E_3 above indicated.

c) It preserves in the local stability features one unstable eigenvalue at E_2 and two unstable eigenvalues at E_3.

For the selection of the controller structure observe that Table I puts in evidence the importance of the second harmonics in the distortion ($\Delta_2/\Delta \simeq 0.18$) so that it can be assumed the controller nonlinearity

$$n_c(y) = \gamma_1 y + \gamma_2 y^2. \tag{56}$$

According to Section IV this control problem is translated into the constrained optimization given by (38) $\div$ (41). Now, it results that the equation $\Phi = 0$ has been already solved predetermining from the analysis just the value A_d corresponding to B_d. Moreover, it is $\Psi = 0$ as indicated by formula (26) for the case where y_d is taken as the average of the chaotic attractor.

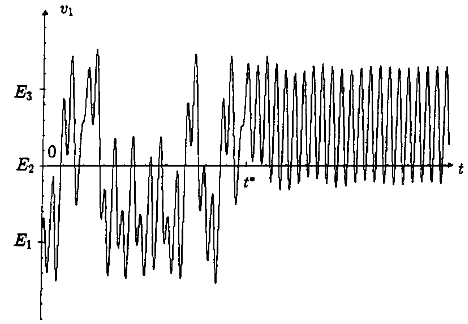

Fig. 5. Dynamics of the unfolded Chua's circuit before and after the control application at $t = t^*$ (large solution, $B_d = 1.105$).

Therefore, by selecting $\lambda_d = 0.5$ the problem reduces to (44) $\div$ (46) in the form

$$\min_{\gamma 2} \left[0.066\gamma_2^2 + 0.024\gamma_2\right], \quad \text{(distortion minimization)}$$

subject to

$$\begin{aligned} &\gamma_1 + 1.675\gamma_2 = 0, \\ &\quad \text{(PLC condition on } y_d) \\ &\gamma_1 < 0.229, \\ &\quad \text{(condition on } E_2 \text{ stability)} \\ &-0.337 < \gamma_1 + 2.974\gamma_2 < 1.530, \\ &\quad \text{(condition on } E_3 \text{ stability).} \end{aligned} \tag{57}$$

By elimination of γ_1 through the equality relation, the final constraint $-0.137 < \gamma_2 < 1.178$ is obtained so that the minimum solution lies on its lower bound. To have a certain stability margin it can be assumed $\gamma_{2d} = -0.1$ which gives the distortion $\Delta_c \simeq 0.023$, a lower value than the original $\Delta \cong 0.049$ (see Table I). It follows $\gamma_{1d} = 0.167$.

Fig. 5 shows the time behavior of the controlled system when the feedback compensator is applied to the chaotic circuit at $t = t^*$. It is evident that the selected controller has destroyed the original system symmetry, so that the derived solution does not stabilize the attractor of negative bias. It can be easily shown that this can be obtained by changing the sign of γ_2 while neglecting the other attractor. Therefore, a switching between $\pm\gamma_2$ suitably entrains the system to one or another of the average chaotic motions.

5.3. *Chaos Control 2*

Again consider the chaotic system of Section 5.2 and assume that it must be controlled in such a way that a) it attains a periodic solution of small amplitude, say $B_d = 0.1$, around the positive bias as

$$y_d(t) = A_d + 0.1 \sin \omega_d t, \tag{58}$$

while the specifications b) and c) are the same as the problem in Section 5.2.

In this case, since the behavior of the original system has been restricted close to the EP $E_3 = 1.49$, its dynamics is really that of a linear circuit due to the piecewise-linear characteristic $\overline{n}$ (see (48)). Then, the role of the compensator is that of leading the controlled system to a quasi Hopf

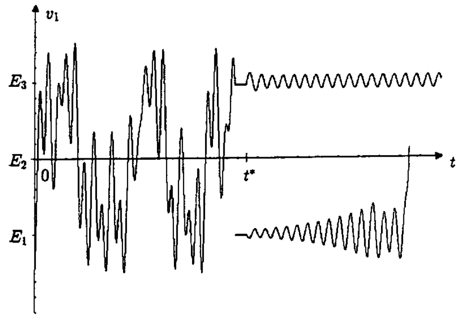

Fig. 6. Dynamics of the unfolded Chua's circuit before and after the control application at $t = t^*$ (small solution, $B_d = 0.1$).

bifurcation and, at the same time, of establishing a given oscillation amplitude. This requires the extension of (56) to the cubic function

$$n_c(y) = \gamma_1 y + \gamma_2 y^2 + \gamma_3 y^3. \quad (59)$$

Now, the bias value clearly results in $A_d = E_3 = 1.49$ and by selecting $\lambda_d = 0.5$ the problem (44) $\div$ (46) reduces to

$$\min_{\gamma_2, \gamma_3} [1.2\gamma_2^2 + 10.4\gamma_2\gamma_3 + 23.3\gamma_3^2]$$

subject to

$$\gamma_1 + 2.97\gamma_2 + 6.64\gamma_3 = -0.337$$

$$\gamma_1 < 0.229$$

$$-0.337 < \gamma_1 + 2.97\gamma_2 + 6.63\gamma_3 < 1.530 \quad (60)$$

where the relations have the meaning of those in (57). The derivation of γ_1 from the equality condition and the substitution in the inequalities allows one to determine the minimum of the distortion cost. As in the preceding case the solution is on the boundary of the admissible $\gamma_2 - \gamma_3$ region: with a few stability margin the values $\gamma_{2d} = -0.002$ and $\gamma_{3d} = -0.001$ can be selected, leading to $\gamma_{1d} = -0.324$. The distortion Δ_c turns out to be very small ($\simeq 10^{-5}$) being the system close to the Hopf bifurcation, while the frequency is $\omega_d \simeq 2.3$.

Fig. 6 shows the time behavior of the control system when it operates from $t = t^*$, after that it has been inizialized in the neighborhood of the EP E_3. In fact, in this case the domain of attraction of the desired solution does not generally contain the trajectories of the original attractor. Fig. 6 also represents that an inizialization close to E_1 does not give rise to a stable periodic solution since the system is not symmetric for the presence of γ_2 in (56). It can be shown that by means of only γ_3 the system could be controlled to two small periodic solutions around E_1 and E_3.

VI. Conclusions

The paper has presented an approach for studying the chaotic behavior of circuits which is founded on a basic elementary model of chaos. By considering an important class of nonlinear dynamical systems, this model is recognized by means of methods based on an approximate harmonic analysis and essentially structural (non-numerical) conditions to detect the chaos onset are derived. This leads to *engineering tools* of reasonable accuracy that have been presented for applications in *analysis* problems as well as in *control* problems where a *stabilization to periodic solutions* is required, preserving the main characteristics of the given chaotic system. Some examples concerning the unfolded Chua's circuit have been also developed to show the applicability of the proposed techniques and the kind of results which can be expected.

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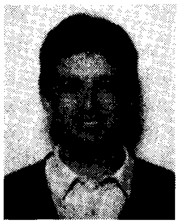

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