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ALMOST INVARIANT SETS IN CHUA'S CIRCUIT*

MICHAEL DELLNITZ† and OLIVER JUNGE‡
Mathematisches Institut, Universität Bayreuth,
D-95440 Bayreuth, Germany

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Recently multilevel subdivision techniques have been introduced in the numerical investigation of complicated dynamical behavior. We illustrate the applicability and efficiency of these methods by a detailed numerical study of Chua's circuit. In particular we will show that there exist two regions in phase space which are *almost invariant* in the sense that typical trajectories stay inside each of these sets on average for quite a long time.

1. Introduction

Suppose that we want to study the complicated dynamical behavior of a given differential equation. Then — due to the nature of chaos — it could be misleading to extract information from long term computations of single trajectories. Rather it would be desirable

(a) to approximate *directly* the topological structure of the underlying invariant set, and then
(b) to derive statistical properties of the dynamical behavior on this set.

For instance, one object which is of interest in this context is the related (*natural*) *invariant measure* (an SBR-measure, say): On the one hand the support of such a measure provides the topological information, and on the other hand, its range yields the statistics of how frequently certain parts of phase space are visited on average by typical trajectories.

However, the invariant measure on its own does not provide all the information on the dynamics which could be of interest. For an illustration of this fact consider two parts in phase space which both have measure 1/2. Then there is equal chance that typical trajectories are observed in each of these regions. On the other hand, nothing can be said about transitions between these two sets. For instance, it may be the case that trajectories are moving rapidly back and forth between them or, in contrast to this, they typically stay in each of them for quite a long period of time before moving to the other one. This observation naturally leads to the question of whether there exist *almost invariant sets*, that is, regions in phase space where typical trajectories stay (on average) for quite a long period of time. The knowledge about the existence of almost invariant sets combined with the information on the natural invariant measure provides an

approximation of the essential dynamics

on invariant sets — this is precisely what we aim for.

Recently subdivision techniques combined with the approximation of the Frobenius–Perron operator have been developed for the numerical extraction of the essential dynamics (see [Dellnitz & Hohmann, 1997; Dellnitz & Junge, 1996] or, for an alternative approach using index theory, [Eidenschink, 1995]). In this article we employ these techniques in order to study in detail the

*The research of the authors is partly supported by the Deutsche Forschungsgemeinschaft under Grant De 448/5-2.
†http://www.uni-bayreuth.de/departments/math/~mdellnitz
‡http://www.uni-bayreuth.de/departments/math/~ojunge

dynamical behavior of *Chua's circuit*,

$$
\begin{aligned}
\dot{x} &= \alpha \left(y - m_0 x - \frac{1}{3} m_1 x^3 \right) \\
\dot{y} &= x - y + z \\
\dot{z} &= -\beta y \,.
\end{aligned}
\tag{1}
$$

This system of differential equations describes the behavior of an autonomous electrical circuit. With regard to the technical background as well as a detailed study of this model by classical methods, the reader is referred to e.g. [Huang *et al.*, 1996]. Throughout our considerations we have fixed the values of the parameters as follows

$$
\alpha = 18, \quad \beta = 33\,,
$$
$$
m_0 = -0.2 \quad \text{and} \quad m_1 = 0.01\,.
$$

In our discussions we will particularly focus on the situation where the underlying system has a $\mathbb{Z}_2$-symmetry, since also Chua's circuit is $\mathbb{Z}_2$-symmetric. In general the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ can be represented by $\{I_n, \kappa\}$ with an orthogonal matrix κ satisfying $\kappa^2 = I_n$ (I_n the identity on $\mathbb{R}^n$). In Chua's circuit $\kappa = -I_3$ since the right hand side in (1) is an odd function in (x, y, z). Correspondingly, $(x(t), y(t), z(t))$ is a solution of (1) if and only if $(-x(t), -y(t), -z(t))$ is a solution.

The bottom line of this article is to approximate the essential dynamics of Chua's circuit. In particular, we compute invariant measures on the closure of two two-dimensional unstable manifolds and show numerically that there exist two almost invariant sets in Chua's circuit which are symmetrically related.

All the algorithms described in the following are integrated into the C++ code

GAIO (**G**lobal **A**nalysis of **I**nvariant **O**bjects).

A link to a detailed description of GAIO can be found on the homepages of the authors.

2. Construction of a Box Covering

The basic technique underlying the approximation of invariant measures and almost invariant sets is the computation of transition probabilities for boxes covering the interesting dynamics in phase space. Hence we begin by describing the numerical multilevel methods we have used to construct such a box covering. We remark that in principle also the classical *cell mapping techniques* could have been used (e.g. [Kreuzer, 1987; Hsu, 1992]). However, our approach has the advantage that the computational effort primarily depends on the complexity of the dynamical behavior rather than on the dimension of phase space.

2.1. *A subdivision algorithm*

The purpose is to approximate invariant sets of discrete dynamical systems of the form

$$
x_{j+1} = f(x_j), \quad j = 0, 1, \ldots\,,
$$

where f is a continuous mapping on $\mathbb{R}^n$. The central object which is approximated by the subdivision algorithm developed by Dellnitz and Hohmann [1997] is the so-called *relative global attractor*,

$$
A_Q = \bigcap_{j \geq 0} f^j(Q)\,, \tag{2}
$$

where $Q \subset \mathbb{R}^n$ is a compact subset. Roughly speaking, the set A_Q should be viewed as the *union of unstable manifolds of invariant objects inside* Q. In particular, A_Q may contain subsets of Q which cannot be approximated by direct simulation.

The subdivision algorithm for the approximation of A_Q generates a sequence $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \ldots$ of finite collections of boxes with the property that for all integers k the set

$$
Q_k = \bigcup_{B \in \mathcal{B}_k} B
$$

is a covering of the relative global attractor under consideration. Moreover the sequence of coverings is constructed in such a way that the diameter of the boxes,

$$
\operatorname{diam}(\mathcal{B}_k) = \max_{B \in \mathcal{B}_k} \operatorname{diam}(B)
$$

converges to zero for $k \to \infty$.

Given an initial collection $\mathcal{B}_0$, one inductively obtains $\mathcal{B}_k$ from $\mathcal{B}_{k-1}$ for $k = 1, 2, \ldots$ in two steps.

(i) *Subdivision:* Construct a new collection $\hat{\mathcal{B}}_k$ such that

$$
\bigcup_{B \in \hat{\mathcal{B}}_k} B = \bigcup_{B \in \mathcal{B}_{k-1}} B
$$

and $\quad \operatorname{diam}(\hat{\mathcal{B}}_k) \leq \theta \operatorname{diam}(\mathcal{B}_{k-1})$

for some $0 < \theta < 1$.

(ii) *Selection:* Define the new collection $\mathcal{B}_k$ by

$$\mathcal{B}_k = \Big\{ B \in \hat{\mathcal{B}}_k : f^{-1}(B) \cap \hat{B} \neq \emptyset \quad \text{for some} \quad \hat{B} \in \hat{\mathcal{B}}_k \Big\} .$$

The following proposition establishes a general convergence property of this algorithm.

Proposition 2.1. *Let A_Q be the global attractor relative to the compact set Q, and let $\mathcal{B}_0$ be a finite collection of closed subsets with $Q_0 = Q$. Then*

$$\lim_{k\to\infty} h(A_Q, Q_k) = 0 ,$$

where we denote by $h(B, C)$ the usual Hausdorff distance between two compact subsets $B, C \subset \mathbb{R}^n$.

For a proof as well as details concerning the implementation of the subdivision algorithm we refer to [Dellnitz & Hohmann, 1997].

2.2. *The approximation of invariant manifolds by continuation*

The subdivision algorithm can be used as a basis to construct coverings of invariant manifolds of hyperbolic objects. In our computations we have used this method, and we now briefly describe a simplified version of it. For a detailed discussion the reader is referred to [Dellnitz & Hohmann, 1996].

Let p be a hyperbolic fixed point and denote by $W^u(p)$ its unstable manifold. As in the subdivision algorithm we fix a (large) compact set $Q \subset \mathbb{R}^n$ containing p, in which we want to approximate part of $W^u(p)$. The idea of the method is as follows: first we apply the subdivision algorithm to a (small) neighborhood C of p, which leads to a box covering of the local unstable manifold $W^u_{\text{loc}}(p)$. (Observe that $W^u_{\text{loc}}(p)$ is the relative global attractor A_C if C is small enough.) Then this covering is extended to the global part of $W^u(p)$ by an action of f on the covering.

To be more precise let $\mathcal{B}_k$ be a covering of the relative global attractor

$$A_C = W^u_{\text{loc}}(p) \cap C$$

obtained by the subdivision algorithm applied to C after k steps. For simplicity assume that the boxes in $\mathcal{B}_k$ are generalized rectangles of identical size and denote by

$$\mathcal{P} \text{ a partition of } Q$$

into a finite collection of disjoint generalized rectangles of this type. For the computation of unstable manifolds we now proceed as follows: define a sequence $\mathcal{C}_0, \mathcal{C}_1, \ldots$ of subsets $\mathcal{C}_j \subset \mathcal{P}$ by

(i) *Initialization:*
$$\mathcal{C}_0 = \mathcal{B}_k .$$

(ii) *Continuation:* For $j = 0, 1, 2, \ldots$ define

$$\mathcal{C}_{j+1} = \{ B \in \mathcal{P} : B \cap f(B') \neq \emptyset \quad \text{for some} \quad B' \in \mathcal{C}_j \}$$

until a j is reached such that $\mathcal{C}_{j+1} = \mathcal{C}_j$.

Intuitively it is clear that the algorithm, as constructed, generates an approximation of part of the unstable manifold $W^u(p)$. Indeed, if we set $W_0 = W^u_{\text{loc}}(p) \cap C$ and define inductively for $j = 0, 1, 2, \ldots$

$$W_{j+1} = f(W_j) \cap Q ,$$

then one can show the following convergence result:

Proposition 2.2 [Dellnitz & Hohmann, 1996]. *For every j the set C_j is a covering of W_j. Moreover, for fixed j, C_j converges to W_j in Hausdorff distance if the number k of subdivision steps in the initialization goes to infinity.*

2.3. *Aspects concerning $\mathbb{Z}_2$-symmetry*

We now outline how to reduce the numerical effort if the underlying dynamical system is $\mathbb{Z}_2$-symmetric. As in the introduction we assume that $\mathbb{Z}_2$ is given by $\{I, \kappa\}$, where κ is an orthogonal matrix with $\kappa^2 = I$. (Recall that Chua's circuit is $\mathbb{Z}_2$-symmetric, where the nontrivial element in $\mathbb{Z}_2$ is given by $\kappa = -I_3$.)

Suppose that the relative global attractor A_Q is invariant under the action of κ, that is,

$$\kappa A_Q = A_Q .$$

Then both the subdivision and the selection steps in the subdivision algorithm can be simplified in an obvious way. For this one has to work with box collections $\mathcal{B}_k$ which are κ-symmetric,

$$B \in \mathcal{B}_k \Longleftrightarrow \kappa B \in \mathcal{B}_k . \tag{3}$$

Indeed in this case one can reduce the numerical effort as well as the memory requirements by a factor of two since in the subdivision algorithm one just needs to take half the number of boxes into account.

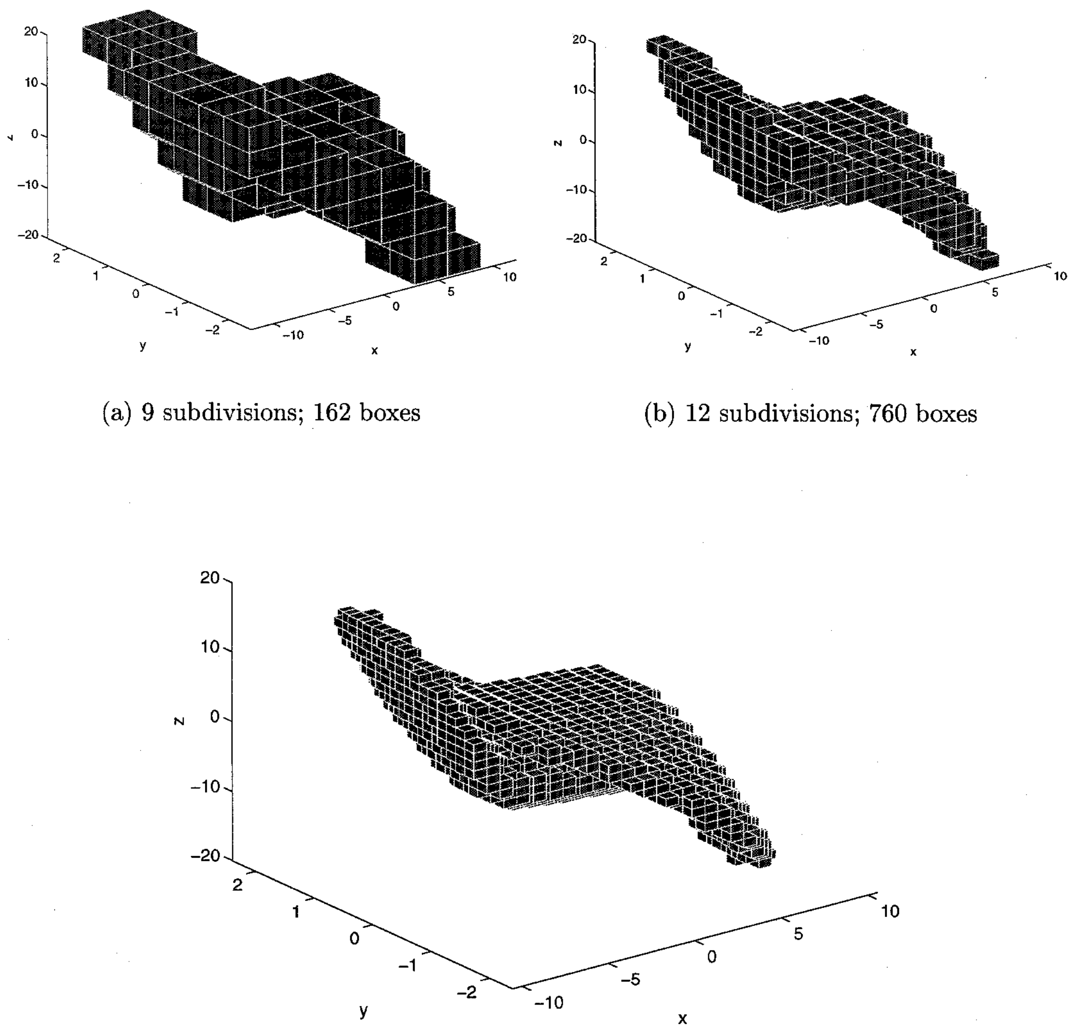
Essentially the same argument can be applied to the continuation method for the approximation of invariant manifolds: One just needs to consider a symmetric Q (i.e. $\kappa Q = Q$) and has to "symmetrize" the $\mathcal{C}_j$'s in each continuation step.

2.4. *Box coverings for Chua's circuit*

In Chua's circuit (1) we have approximated the union of the two two-dimensional unstable manifolds of the steady state solutions $\pm(7.746,\ 0,\ -7.746)$ inside the box $Q = [-12,\ 12] \times [-2.5,\ 2.5] \times [-20,\ 20]$. Note that $Q = -Q$ and therefore we can use the underlying symmetry of Chua's circuit. We have computed the time-one-map f numerically using a fourth order Runge–Kutta method with constant step length $h = 0.1$. Four different box coverings are shown in Figs. 1 and 2. Already after 12 steps in the subdivision algorithm the shape of the invariant set given by the closure of the unstable manifolds becomes apparent.

3. Approximation of the Essential Dynamics

In the previous section we have seen how to construct a box covering of the relevant dynamical behavior in phase space. In other words we have the



(a) 9 subdivisions; 162 boxes

(b) 12 subdivisions; 760 boxes

Fig. 1. Three coverings of an attracting set A for Chua's circuit. The set A is the closure of the union of the two-dimensional unstable manifolds of the two nonsymmetric steady states.

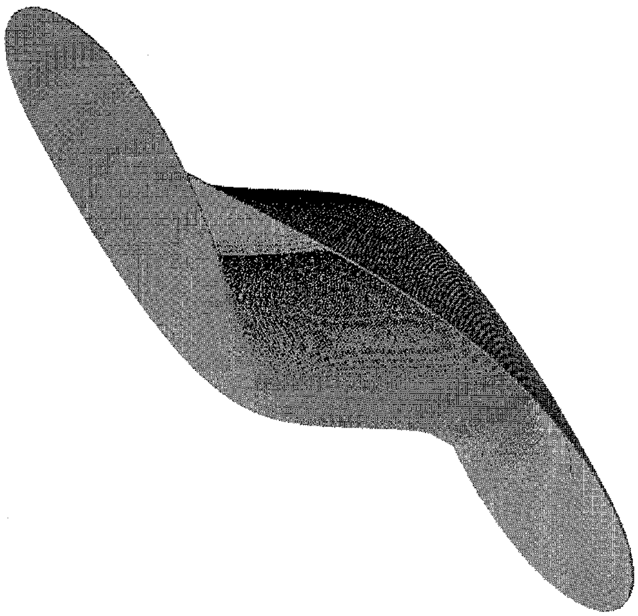


Fig. 2. Covering of an attracting set for Chua's circuit (27 subdivisions, 877346 boxes).

topological information needed and it remains to approximate the dynamical behavior on this object. This is done by

(a) an approximation of the natural invariant measure supported on the invariant set (e.g. an *SBR-measure*, if it would exist), and
(b) an identification of subsets of the relative global attractor which are *almost invariant*.

The main purpose of this article is to apply the related numerical techniques to Chua's circuit, and hence we will briefly recall them from [Dellnitz & Junge, 1996].

3.1. *Discretization of the Frobenius-Perron operator*

The crucial observation is that the calculation of invariant measures can be viewed as a fixed point problem. Let $\mathcal{M}$ be the set of probability measures on $\mathbb{R}^n$ with respect to the Borel σ-algebra. Then $\mu \in \mathcal{M}$ is invariant if and only if it is a fixed point of the *Frobenius–Perron operator* $P : \mathcal{M} \to \mathcal{M}$,

$$(P\mu)(B) = \mu(f^{-1}(B)) \quad \text{for all measurable } B \subset \mathbb{R}^n . \tag{4}$$

To discretize the operator $P : \mathcal{M} \to \mathcal{M}$ we replace $\mathcal{M}$ by a finite dimensional set $\mathcal{M}_k$: let $B_i \in \mathcal{B}_k$, $i = 1, \ldots, N$, denote the boxes in the covering obtained after k steps in the subdivision algorithm. We choose $\mathcal{M}_k$ to be the set of "discrete probability measures" on $\mathcal{B}_k$, that is,

$$\mathcal{M}_k = \left\{ u : \mathcal{B}_k \to [0,\, 1] \,\middle|\, \sum_{i=1}^{N} u(B_i) = 1 \right\} .$$

Then the discretized Frobenius–Perron operator $P_k : \mathcal{M}_k \to \mathcal{M}_k$ is given by

$$v = P_k u, \quad v(B_i) = \sum_{j=1}^{N} \frac{m(f^{-1}(B_i) \cap B_j)}{m(B_j)} u(B_j) , \quad i = 1, \ldots, N , \tag{5}$$

where m denotes Lebesgue measure. Now a fixed point $u = P_k u$ of P_k provides an approximation to an invariant measure of f. Observe that P_k is represented by a stochastic matrix.

Remark 3.1. For the mathematically precise statement on the convergence of this method one would have to introduce the concept of *small random perturbations*. The reason is that this allows one to use a result of Kifer [1986] on the convergence of invariant measures in the perturbed systems to the SBR-measure. The reader is referred to [Dellnitz & Junge, 1996] for the rigorous mathematical treatment. (See also [Kifer, 1996].)

3.2. *Aspects concerning $\mathbb{Z}_2$-symmetry*

If the box covering is chosen in a symmetric way [see (3)] then the numerical effort for the computation of the discretized Frobenius–Perron operator in (5) can be reduced by a factor of two. Indeed, since κ is orthogonal

$$\frac{m(f^{-1}(B_i) \cap B_j)}{m(B_j)} = \frac{m(f^{-1}(\kappa B_i) \cap \kappa B_j)}{m(\kappa B_j)} \quad \text{for all } i,\, j \tag{6}$$

and therefore just half of the entries of P_k have to be computed. This also reduces the memory requirements — observe that typically the matrix P_k is extremely sparse.

By (6) one can easily verify that the discretized Frobenius–Perron operator commutes with the action of κ, that is

$$\kappa P_k = P_k \kappa . \tag{7}$$

(Here we have abused notation and denoted by κ also the matrix which is permuting the indices of symmetrically placed boxes.) It follows that an eigenvector v of P_k corresponding to a simple eigenvalue is either

$$\textit{symmetric}\ (\kappa v = v)\ \text{or}\ \textit{anti-symmetric}\ (\kappa v = -v)\,.$$

To see this use (7) to obtain

$$P_k v = \lambda v \Longrightarrow P_k(\kappa v) = \kappa P_k v = \kappa \lambda v = \lambda(\kappa v)\,,$$

and therefore κv must be a real multiple of v. Finally note that κ is orthogonal.

3.3. *The natural invariant measure μ for Chua's circuit*

Suppose that there exists a unique natural invariant measure μ with support on the closure of the two two-dimensional unstable manifolds considered in Sec. 2. For the approximation of μ we have to compute the eigenvector u of the discretized Frobenius–Perron operator corresponding to the eigenvalue 1. By the uniqueness of the invariant measure the support of μ has to be symmetric with respect to $\kappa = -I_3$. Hence we can use the $\mathbb{Z}_2$-symmetry in the computations as indicated above. In particular we know that u has to be symmetric since its entries are non-negative by standard results on eigenvectors of stochastic matrices.

For the visualization we have used two different techniques which have been developed in [Dellnitz *et al.*, 1997]. In Fig. 3 we cut the invariant set by a plane and show the invariant measure along the intersection for six different cuts. Here also the complicated topological structure of the unstable manifolds becomes apparent. The invariant measure on the entire invariant set is illustrated in Fig. 4: Yellow colored boxes indicate high density, and hence typical trajectories can frequently be observed in these regions. On the other hand, typical trajectories will hardly visit the regions which are colored red and probably never enter the green areas.

Remark 3.2. An animation with a "moving cutting plane" can be found on the homepages of the authors.

3.4. *Almost invariant sets in the presence of $\mathbb{Z}_2$-symmetry*

The natural invariant measure μ provides the information where, on average, typical trajectories will be observed in the given system. However, the invariant measure does not provide all the information about the actual dynamical behavior. For instance, if there are two parts in state space which both have measure 1/2 then chances are equal that trajectories are observed in one of those regions. But it is not clear whether trajectories are moving quickly back and forth between these sets or stay in each of them for quite a long time before moving to the other one. In the latter case the components should be viewed as being almost invariant: Let $\rho \in \mathcal{M}$ be a probability measure. We say that the set B is *δ-almost invariant with respect to ρ* if

$$\frac{\rho(f^{-1}(B) \cap B)}{\rho(B)} = \delta\,.$$

Thus, δ can be viewed as the probability that points in B are mapped into B under f. In particular, if B is an invariant set, that is $f^{-1}(B) = B$, then $\delta = 1$. The bottom line of this section is to relate the size of δ to the size of certain eigenvalues of the Frobenius–Perron operator P.

Suppose that $\lambda \neq 1$ is a real eigenvalue of P with corresponding real valued eigenmeasure $\nu \in \mathcal{M}_{\mathbb{C}}$, that is,

$$P\nu = \lambda\nu\,,$$

where ν is scaled so that $|\nu| \in \mathcal{M}$. (We denote by $\mathcal{M}_{\mathbb{C}}$ the set of bounded complex valued measures.)

Remark 3.3. Obviously, ν itself cannot be a probability measure — in fact, it is not difficult to see that $\nu(\mathbb{R}^3) = 0$. However, if the eigenvalue λ is close to one it is reasonable to assume that the probability measure $|\nu|$ is close to the invariant measure μ of the system. To see this consider the "unperturbed case" where $\lambda = 1$ is a double eigenvalue. Then there are two disjoint invariant sets A_1, A_2 and one can choose independent eigenmeasures μ and ν such that $|\nu| = \mu$. In particular, for this choice one has $\nu(B_1) = \mu(B_1)$ and $\nu(B_2) = -\mu(B_2)$ for all $B_j \subset A_j$.

Recall that Chua's circuit is κ-symmetric where $\kappa = -I_3$. It is easy to see that the Frobenius–Perron operator P commutes with κ, where the action of κ on a measure ρ is given by

$$(\kappa^*\rho)(D) = \rho(\kappa D) \qquad \text{for all measurable } D \subset \mathbb{R}^3\,.$$

Since by assumption λ is a simple eigenvalue it follows as in the finite dimensional case that the eigenmeasure ν is either

$$\text{symmetric } (\kappa^*\nu = \nu) \text{ or anti-symmetric } (\kappa^*\nu = -\nu)\,.$$

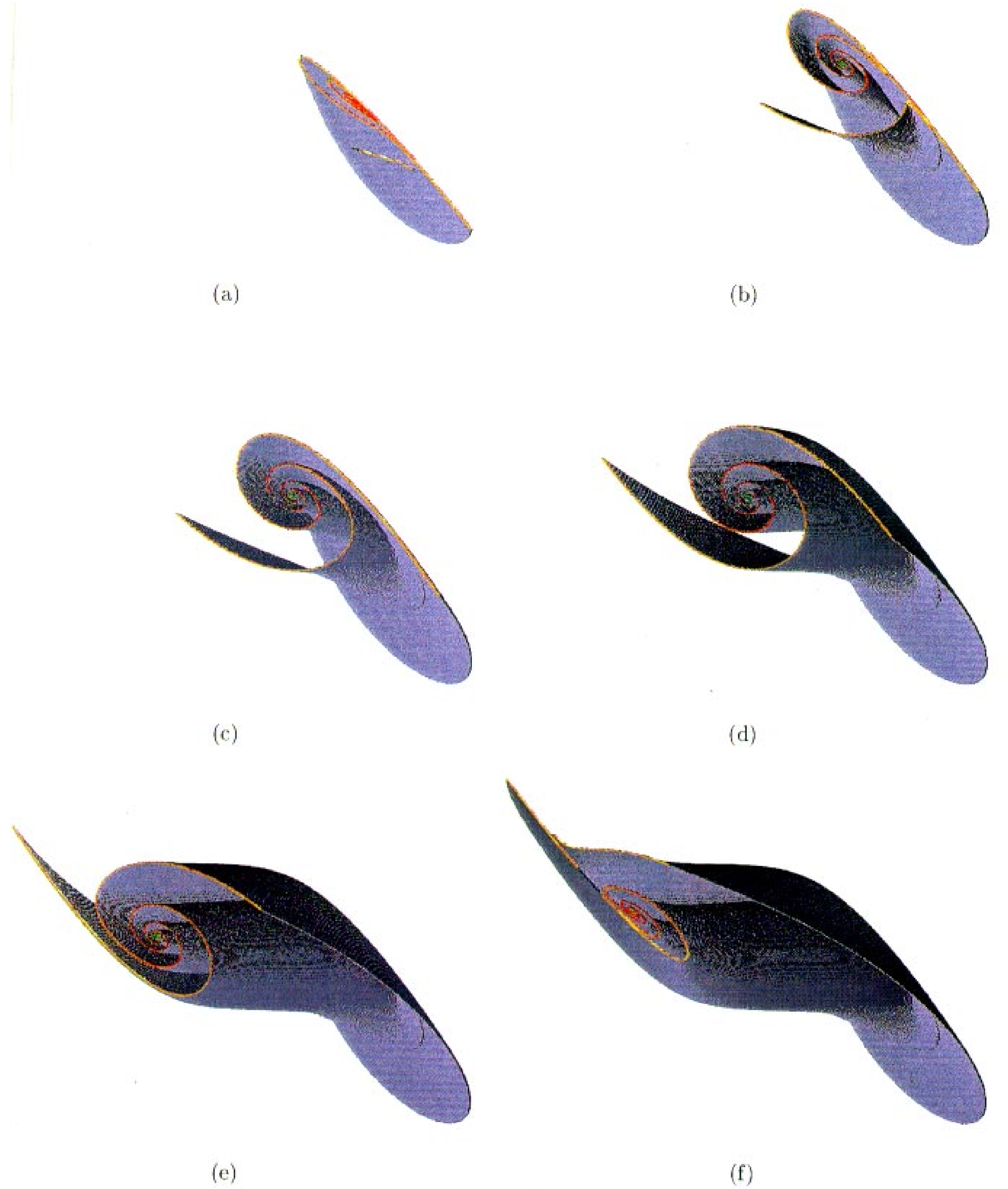


Fig. 3. Cutting planes illustrating the structure of the attracting set in Chua's circuit as well as the natural invariant measure (27 subdivisions, 877346 boxes). In the cutting planes the density ranges from green (low density) $\longrightarrow$ red $\longrightarrow$ orange $\longrightarrow$ yellow (high density).

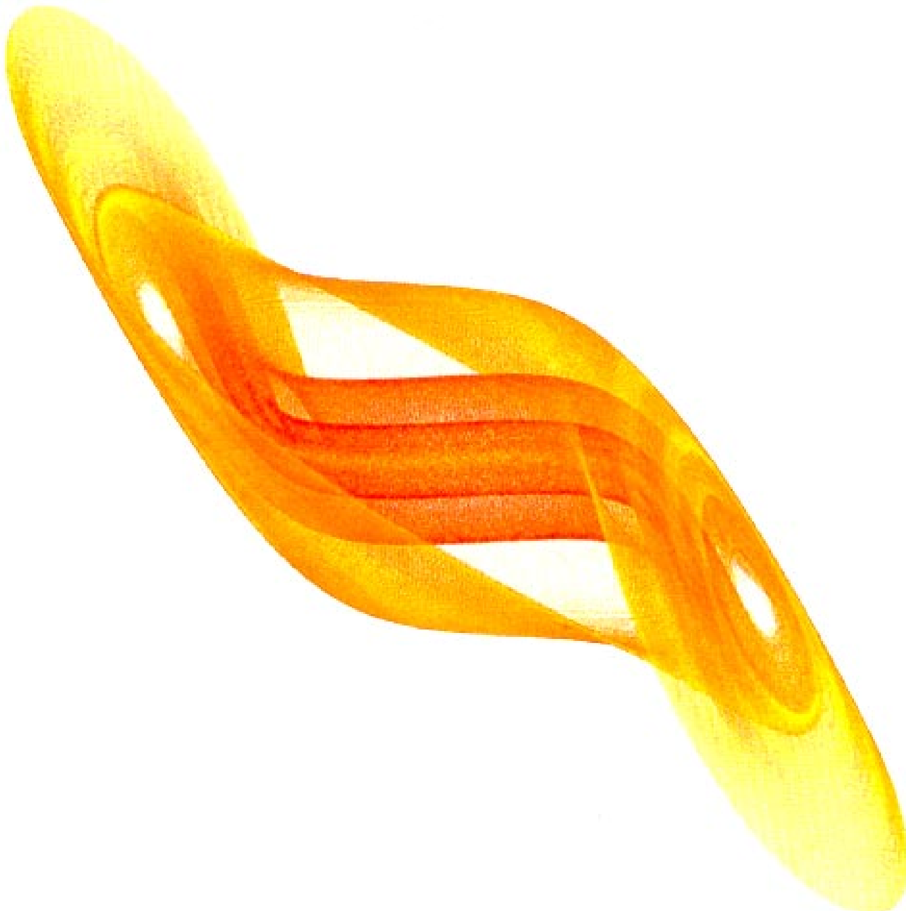


Fig. 4. The natural invariant measure in Chua's circuit (27 subdivisions, 877346 boxes), coloring scheme as in Fig. 3.

In any case the probability measure $|\nu|$ is symmetric. From now on we assume that ν is antisymmetric since this is the situation we are concerned with in Chua's circuit. Then the related attracting invariant set A inside $Q = [-12, 12] \times [-2.5, 2.5] \times [-20, 20]$ can be decomposed as

$$A = B \cup \kappa B, \quad \text{where } \nu(B) = \frac{1}{2} \text{ and } \nu(\kappa B) = -\frac{1}{2}.$$

To see this recall that $\nu(A) = 0$ (Remark 3.3).

In the following proposition we relate the size of the eigenvalue λ to the probability δ occurring in the definition of an almost invariant set. A proof in the more general context of small random perturbations can be found in [Dellnitz & Junge, 1996].

Proposition 3.4. *The set κB is δ-almost invariant with respect to $|\nu|$ if and only if B is δ-almost invariant. Moreover δ is given by*

$$\delta = \frac{\lambda + 1}{2}. \tag{8}$$

Proof. Let B be δ-almost invariant with respect to $|\nu|$. Then by definition

$$|\nu|(f^{-1}(B) \cap B) = \delta|\nu|(B) = \frac{\delta}{2}.$$

Using the invariance of A we obtain

$$-\frac{1}{2} = \nu(\kappa B) = \nu(f^{-1}(B) \cap \kappa B) + \nu(f^{-1}(\kappa B) \cap \kappa B).$$

Since $\nu = |\nu|$ on B it follows that

$$\begin{aligned} \nu(f^{-1}(B) \cap \kappa B) &= -\nu(f^{-1}(\kappa B) \cap \kappa B) - \frac{1}{2} \\ &= \nu(f^{-1}(B) \cap B) - \frac{1}{2} \\ &= \frac{1}{2}(\delta - 1). \end{aligned}$$

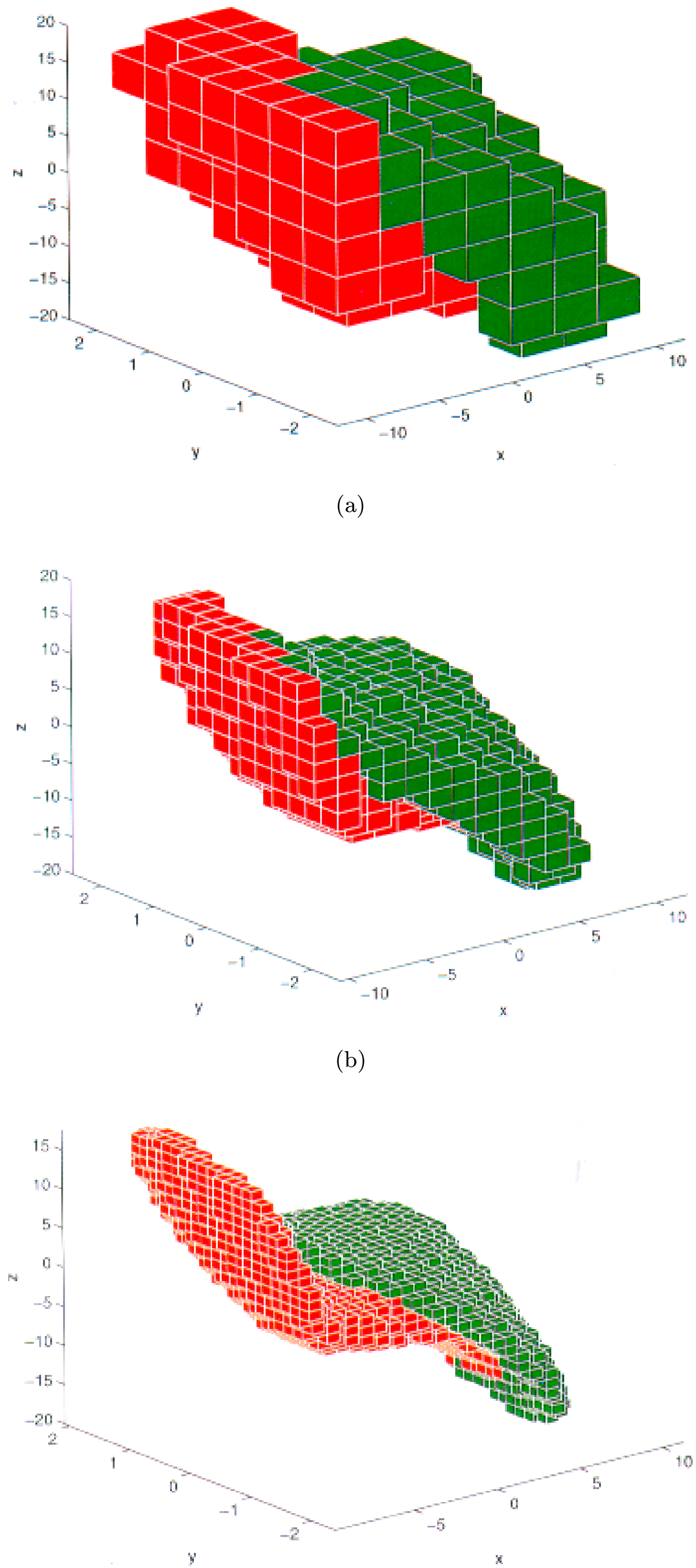


Fig. 5. Box coverings of two almost invariant sets in Chua's circuit. (a) 9 subdivisions, (b) 12 subdivisions, (c) 15 subdivisions.

Table 1. Largest real eigenvalues of $P_{T,k}$.

	$T = 0.1$		$T = 0.2$		$T = 0.5$	
k	λ_0	λ_1	λ_0	$\lambda_1^{1/2}$	λ_0	$\lambda_1^{1/5}$
9	0.961	0.897	0.961	0.879	0.973	0.926
12	0.990	0.890	0.993	0.866	0.998	0.912
15	0.998	0.930	1	0.938	0.999	0.946
18	1	0.945	1	0.947	0.999	0.948

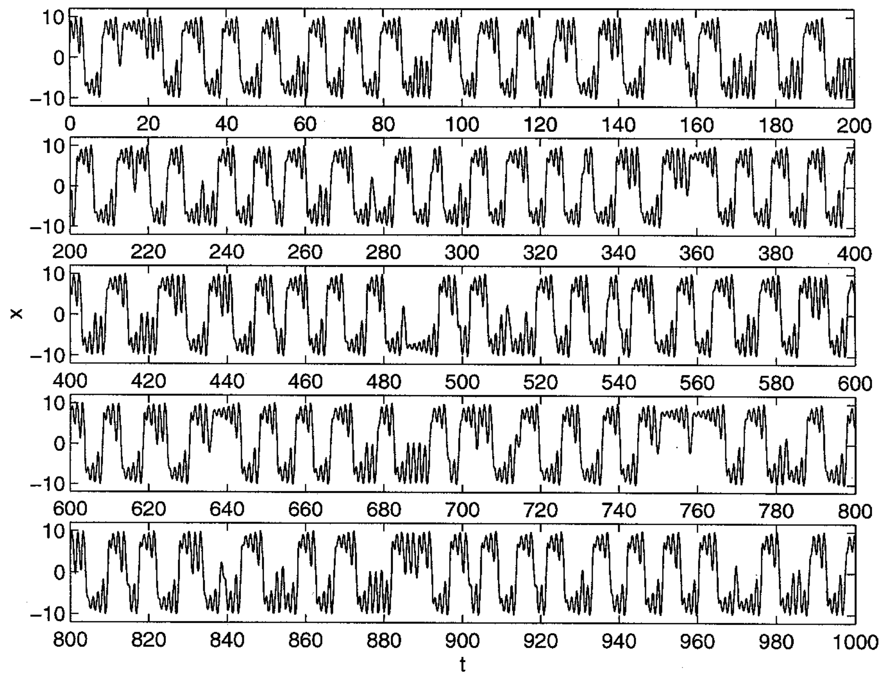


Fig. 6. A simulation of Chua's circuit: x versus time t in seconds.

Finally the fact that λ is an eigenvalue with eigenmeasure ν implies that

$$\begin{aligned}\frac{\lambda}{2} &= \lambda\nu(B) = \nu(f^{-1}(B)) \\ &= \nu(f^{-1}(B)\cap B) \\ &\quad + \nu(f^{-1}(B)\cap \kappa B) = \frac{1}{2}(2\delta - 1)\,.\end{aligned}$$

Now (8) immediately follows. ■

3.5. *Two almost invariant sets in Chua's circuit*

We now apply the results of the previous paragraph and compute eigenvectors corresponding to real eigenvalues of the discretized Frobenius–Perron operator P_k which are close to one.

Again we compute box-coverings $\mathcal{B}_k$ for $k = 9, 12, 15, 18$ by the methods described in Sec. 2. In order to obtain reliable numerical estimates we use time-T-maps with $T = 0.1$, 0.2 and 0.5. In Table 1 we show the approximations λ_0 to the eigenvalue 1 of the corresponding discretized Frobenius–Perron operator $P_{T,k}$ as well as the second largest real eigenvalue λ_1. Observe that we need to consider $\lambda_1^{0.1/T}$ for a comparison of the different results.

Under the hypothesis that λ_1 is quite accurate if $\lambda_0 \approx 1$ the numbers in Table 1 indicate that $P_{T,k}$ has a real eigenvalue λ which is close to 0.95 for $T = 0.1$. As mentioned before the corresponding eigenmeasure ρ is antisymmetric, and therefore an application of Proposition 3.4 leads to $\delta = 0.975$ for a set B with $\rho(B) = 1/2$. Thus, $\delta = 0.975$ may be interpreted as the $|\rho|$-probability for the system to stay in B for 0.1 seconds if the trajectory starts in B at a randomly chosen point.

Table 2. Approximations of the probability to stay inside B for T seconds.

$T = 0.1$	$T = 1$	$T = 5$	$T = 10$	$T = 20$
$\delta = 0.975$	$\delta^{10} = 0.776$	$\delta^{50} = 0.282$	$\delta^{100} = 0.080$	$\delta^{200} = 0.006$

In Fig. 5 we visualize approximations of the almost invariant sets B and κB. More precisely we show in red resp. green the positive resp. negative components of eigenvectors $v_{T,k}$ corresponding to the eigenvalue λ_1 of $P_{T,k}$ for $T = 0.2$ and $k = 9$, 12, 15. Since we want to avoid inaccuracies induced by the numerical approximation we show in each subfigure only those boxes B_j in the covering with $|v_{T,k}(B_j)| > 10^{-3}$.

Finally in Table 2 we compute approximations of the probability to stay within the almost invariant set B for a certain period of time by a computation of powers of δ. Obviously the numbers in Table 2 are dubious for small T. The reason is that trajectories do not enter the set B at random locations, and therefore the interpretation of δ fails in this situation. On the other hand, due to the chaotic behavior of the system the numbers are reasonable for larger T. In fact, Table 2 indicates that the event of the system staying in B for more than 20 seconds is very rare and this can indeed be confirmed by a direct simulation of Chua's circuit (see Fig. 6): Roughly the sign of $x(t)$ indicates whether the system is inside B or κB, and therefore the system stays just once inside one of these components for more than 20 seconds (namely for t around 760).

Acknowledgment

For Figs. 2, 3 and 4 we have used the visualization platform GRAPE which has been developed at the SFB 256 at the University of Bonn and at the Institute for Applied Mathematics at the University of Freiburg.

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