Cycling Chaos

Michael Dellnitz, Michael Field, Martin Golubitsky, Andreas Hohmann, and Jun Ma

Abstract— **Saddle connections between equilibria can occur structurally stable in systems with symmetry, and these saddle connections can cycle so that a given equilibrium is connected to itself by a sequence of connections. These cycles provide a way of generating intermittency, as a trajectory will spend some time near each saddle before quickly moving to the next saddle. Guckenheime and Holmes showed that cycles of saddle connections can appear via bifurcation. In this paper, we show numerically that the equilibria in the Guckenheimer-Holmes example can be replaced by chaotic sets, such as those that appear in a Chua circuit or a Lorenz attractor. Consequently, there are trajectories that behave chaotically, but where the spatial location of the chaos cycles. We call this phenomenon** *cycling chaos.*

I. Introduction

ONE of the characteristic and distinguishing features of symmetric systems of differential equations is the existence of structurally stable saddle connections that would be regarded as highly degenerate in the absence of symmetry (see [3]). Guckenheimer and Holmes [8] (abstracting a model for rotating convection developed by Busse and Clever [1]) showed that it was possible for structurally stable, asymptotically stable, cycles of saddle connections to be created by bifurcation at the loss of stability of a group invariant equilibrium in a symmetric system. The resulting heteroclinic cycle consists of a finite set of saddle points connected by trajectories. Trajectories which approach the cycle remain for long periods near each equilibrium before making a rapid transition to a neighborhood of the next equilibrium. This feature of an asymptotically stable heteroclinic cycle gives a simple mathematical model for intermittence. (A number of other examples are now known where heteroclinic cycles between equilibria and limit cycles are created at a bifurcation. See, for example, Melbourne *et al.* [9], Field and Swift [5], and Field and Richardson [6]).

In this paper, we observe that the equilibria in the Guckenheimer-Holmes heteroclinic cycle may be replaced by chaotic sets. In this way, we present a phenomenological mathematical model for (spatially) cycling (temporal) chaos.

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II. Symmetrically Coupled Cells

Golubitsky *et al.* [7] observed that, with an appropriate choice of coupling, the Guckenheimer-Holmes example can be viewed as a symmetrically coupled system of cells (see also Dionne *et al.* [2]). For our purposes, we regard a cell as a k-dimensional system of ordinary differential equations (ODE's)

$$\dot{x} = f(x)$$

where $x \in \mathbf{R}^k$. Thus, in the Guckenheimer-Holmes system, $k = 1$ and there are three identical cells coupled in a directed ring. Generally, let x_1, x_2, x_3 be in $\mathbf{R}^k$. We consider a system of three coupled cells of the form

$$\begin{aligned}\dot{x}_1 &= f(x_1) + h(x_3, x_1)\\ \dot{x}_2 &= f(x_2) + h(x_1, x_2)\\ \dot{x}_3 &= f(x_3) + h(x_2, x_3).\end{aligned} \tag{1}$$

We identify two types of symmetry in a coupled cell system of this type: global and local. The global symmetries are dictated by the pattern of coupling. In (1), the global symmetry group is $\mathbf{Z}_3$ and is generated by the cyclic permutation

$$(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1).$$

Local symmetries are symmetries of f. Thus, a linear transformation σ of $\mathbf{R}^k$ is a local symmetry if

$$f(\sigma x) = \sigma f(x), \quad (x \in \mathbf{R}^k).$$

In the Guckenheimer-Holmes system, the local symmetry group is $\mathbf{Z}_2$ and is generated by $\sigma x = -x$. Moreover, in this system, local symmetries of individual cells are symmetries of the complete system (1). That is, for all local symmetries σ we have

$$\begin{aligned}h(\sigma y, x) &= h(y, x)\\ h(y, \sigma x) &= \sigma h(y, x).\end{aligned}$$

Following Golubitsky *et al.* [7], we call this type of coupling wreath product coupling. Viewed in this way, the Guckenheimer and Holmes [8] system has coupling term given by the cubic polynomial

$$h(y, x) = \gamma |y|^2 x$$

where $\gamma \in \mathbf{R}$ represents the strength of the coupling.

The internal cell dynamics in the Guckenheimer and Holmes [8] system are governed by the pitchfork bifurcation

$$f(x) = \lambda x - x^3$$

which is consistent with the internal symmetry. As λ varies from negative to positive through zero, a bifurcation from

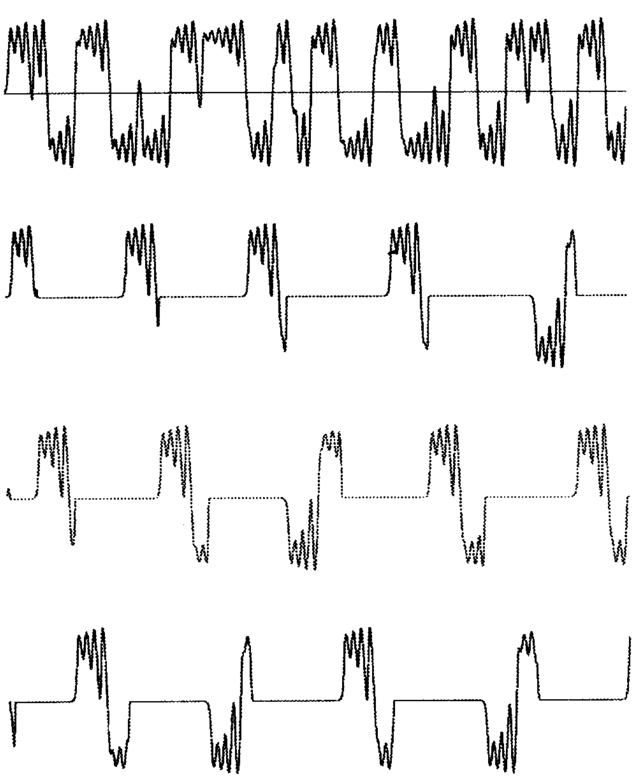

Fig. 1. Chua circuit equations: In the first row, we display the temporal evolution of a single cell without any coupling. In the following three rows, the time series for each of the three cells is shown when the coupling strength is chosen to be $\gamma = -2$. Parameter values: $\alpha = 18$, $\beta = 33.136$, $m_0 = -0.230769$, $m_1 = 0.0123077$. Initial conditions: $x_1(0) = (0.01, 0.1, -0.2)$, $x_2(0) = (0.24, 0.34, -0.01)$, $x_3(0) = (0.2, -0.3, 0.1)$.

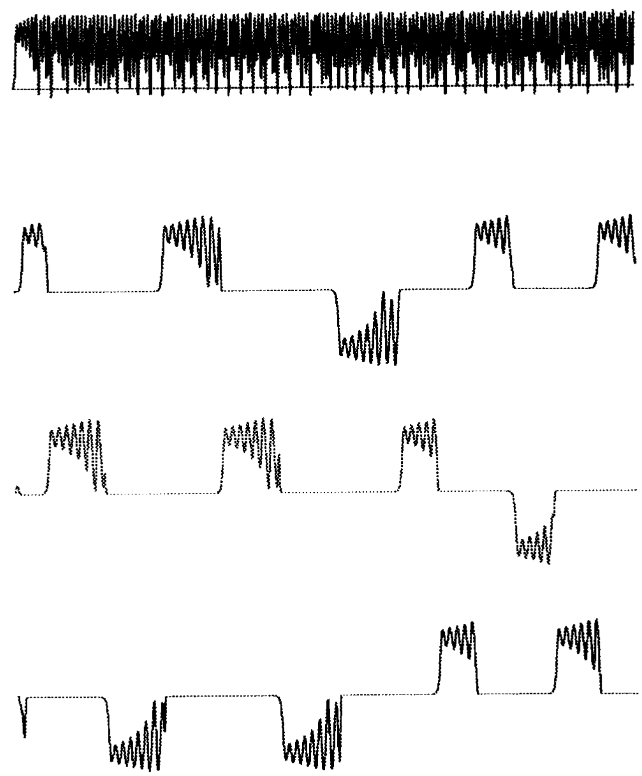

Fig. 2. Chua circuit equations: In the first row, we display the temporal evolution of a single cell without any coupling. In the following three rows, the time series for each of the three cells is shown when the coupling strength is chosen to be $\gamma = -2$. Parameter values: $\alpha = 15$, $\beta = 33.136$, $m_0 = -0.230769$, $m_1 = 0.0123077$. Initial conditions: $x_1(0) = (0.01, 0.1, -0.2)$, $x_2(0) = (0.24, 0.34, -0.01)$, $x_3(0) = (0.2, -0.3, 0.1)$.

the trivial equilibrium ($x = 0$) to nontrivial equilibria ($x = \pm\sqrt{\lambda}$) occurs, and these bifurcating equilibria are stable in the internal cell dynamics. Guckenheimer and Holmes [8] show that when the strength of the coupling is large and negative ($\gamma \ll 0$), an asymptotically stable heteroclinic cycle connecting these bifurcated equilibria exists. The connection between the equilibria in cell 1 to the equilibria in cell 2 occurs through a saddle-sink connection in the x_1x_2-plane (which is forced by the internal symmetry to be an invariant plane for the dynamics). The global permutation symmetry guarantees connections in both the x_2x_3-plane and the x_3x_1-plane.

III. Cycling Chaos

It turns out that the intermittent cycling of the global dynamics does not depend in an essential way on the nature of the internal dynamics, provided that f satisfies some mild restrictions (for example, the origin and infinity are repellors). Detailed mathematical analysis and generalizations will appear in Field *et al.* [4]. In this paper, we illustrate this observation by numerical simulation of the internal dynamics in a three-cell system. Our first two examples have internal dynamics defined by (a modified) Chua circuit

$$f(y_1, y_2, y_3) = (\alpha(y_2 - m_0 y_1 - \frac{m_1}{3} y_1^3), y_1 - y_2 + y_3, -\beta y_3)$$

where the internal variable $x = (y_1, y_2, y_3)$ and α, β, m_0, m_1 are constants. Our third example has internal dynamics governed by the Lorenz equations

$$f(y_1, y_2, y_3) = (\sigma(y_2 - y_1), \rho y_1 - y_2 - y_1 y_3, -\beta y_3 + y_1 y_2)$$

where σ, ρ, β are constants.

In Fig. 1, we choose parameter values so that the internal dynamics is a double scroll attractor. We present a time series of a single cell in the first time history. The following three time evolutions in Fig. 1 show the temporal behavior for cell 1, cell 2, and cell 3, respectively. From this figure, we see that when one of the cells is active and performing the double scroll dynamics—say cell 1—the others (cell 2 and cell 3) are quiescent (near zero). After a while, cell 1 becomes quiescent while cell 2 becomes active, and the transition time during which the cells interchange states is very short. The process then repeats with cell 2 and cell 3 interchanging active and quiescent states. Indeed, the process cycles forever, just as in the Guckenheimer-Holmes heteroclinic cycle, but now producing cycling chaos.

In the second example, we choose parameter values in the Chua circuit which yield an asymmetric chaotic attractor. The internal symmetry forces a second conjugate attractor, and which attractor is actually observed depends on the initial conditions in the individual cell. In Fig. 2, we illustrate this asymmetry. For one choice of initial conditions, the internal

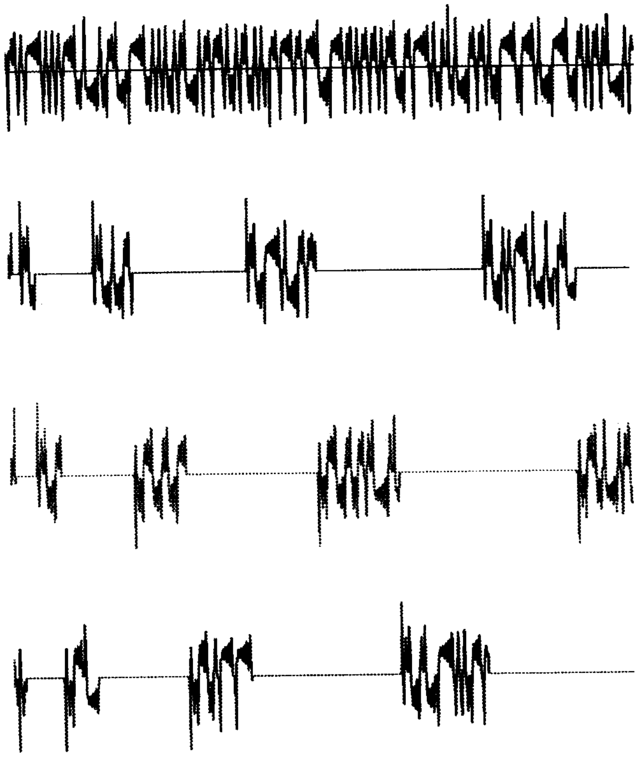

Fig. 3. Lorenz equations: In the first row, we display the temporal evolution of a single cell without any coupling. In the following three rows, the time series for each of the three cells is shown when the coupling strength is chosen to be $\gamma = -0.025$. Parameter values: $\sigma = 15$, $\rho = 58$, $\beta = 2.4$. Initial conditions: $x_1(0) = (10, -11, 30)$, $x_2(0) = (10, -13, 20)$, $x_3(0) = (10, -12, 30)$.

cell dynamics always has $y_1 > 0$; a different choice of initial conditions leads to an attractor where $y_1 < 0$ for all time. When we simulate the coupled cell system, we get the same cycling chaos but when a given cell—say cell 2—becomes active it chooses "randomly" which of the conjugate attractors ($y_1 < 0$ or $y_1 > 0$) it will track.

Finally, in Fig. 3, we illustrate the phenomenon of cycling chaos when the internal dynamics is given by the Lorenz system. Note that the cycling exists even though the internal symmetry in the two examples is different. In the Chua circuit $\sigma(x) = -x$ while in the Lorenz equation $\sigma(y_1, y_2, y_3) = (-y_1, -y_2, y_3)$.

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