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ON BIFURCATIONS LEADING TO CHAOS IN CHUA'S CIRCUIT

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Bifurcations and the structure of limit sets are studied for a three-dimensional Chua's circuit system with a cubic nonlinearity. On the base of both computer simulations and theoretical results a model map is proposed which allows one to follow the evolution in the phase space from a simple (Morse-Smale) structure to chaos. It is established that the appearance of a complex, multistructural set of double-scroll type is stimulated by the presence of a heteroclinic orbit of intersection of the unstable manifold of a saddle periodic orbit and unstable manifold of an equilibrium state of saddle-focus type.

1. Introduction

Nowadays, there exists a specific interest in the study of dynamical systems with chaotic behavior. The stimulating factor is that the phenomenon of dynamical chaos is typical in some sense and it is found in many applications. Usually, there appear difficulties with the answer on the following questions: which limit sets the stochastic behavior is connected with and what is the mechanism of transition from regular oscillations to the stochastic regime? Rigorously, the mathematical image of stochastic oscillations may be an attractive transitive limit set consisting of unstable orbits, that is a strange attractor. The well-known example is Lorenz attractor [Afraimovich *et al.*, 1977, 1983].

However, very often the appearance of chaos is connected with more complicated mathematical object: with a limit set containing nontrivial hyperbolic sets as well as stable periodic orbits (the latter may be invisible due to small absorbing domains and large periods). Such limit sets may generate stochastic oscillations because of the presence of perturbations stimulated by the inevitable presence of noise in experiments and by round-off errors in computer modeling. These limit sets are known as *quasiattractors* [Afraimovich & Shil'nikov, 1983].

A typical example of a quasiattractor is the so-called Rossler attractor arising after a period-doubling cascade. Another example is the spiral attractor which, somehow, is a union of the Rossler attractor and the unstable limit set near a homoclinic loop of a saddle-focus [Shil'nikov, 1991]. Also, if two spiral quasiattractors unite including the saddle-focus together with its invariant unstable manifold, then a more complicated set arise which is called *double-scroll* [Chua *et al.*, 1986].

The quasiattractors listed above are not abstract mathematical objects but they correspond to real processes visible in experiments as well as in computer simulations with, for instance, the well-known Chua's circuit (Fig. 1) [Madan, 1993]. The study of this circuit was mainly carried out for the case of piecewise linear approximation of the nonlinear element [Chua *et al.*, 1986, Chua & Lin, 1991; George, 1986]. Note that the interest to the

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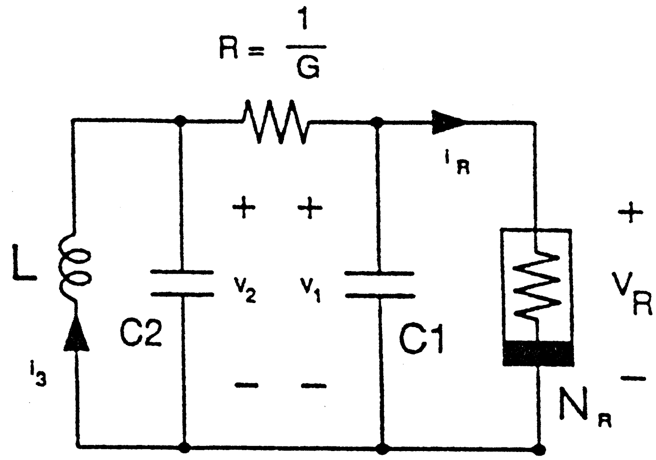


Fig. 1. Circuit scheme for Chua's circuit.

piecewise linear representation is, apparently, connected not with the technical aspect of the problem but with the idea of the applicability of analytical methods for this case. It happened, nevertheless, that the complexity of the analytical expressions arising here is too high, their immediate analysis is quite difficult and cannot actually be done without use of numerical methods. In fact, the direct computer simulations of the differential equations has appeared to be more effective. Note also that the piecewise linear approximation does not allow one to use the full capacity of the methods and results of the bifurcation theory developed mostly for smooth dynamical systems. All this were the reasons why the characteristic of the nonlinear element was modeled in [Freire *et al.*, 1993; Khibnic *et al.*, 1993] by a cubic polynomial which retains the main geometrical features of the piecewise linear approximation. This choice provided the possibility of studying local bifurcations by analytical methods and, then, by numerical simulation, to show the presence of global bifurcations, in particular, those which indicate chaotic dynamics.

The scope of the present paper is to study the main bifurcations and the structure of limit sets for the following three-dimensional system.

$$\dot{x} = \beta(g(y-x) - h(x)), \quad \dot{y} = g(x-y) + z, \quad \dot{z} = -y \tag{1}$$

where β, g, α are positive parameters describing the aforementioned electronic circuit for the cubical $h(x) = \alpha x(x^2 - 1)$ approximation of the nonlinear element. The parameters β and g are connected with the physical parameters of the circuit as follows: $\beta = C_1/C_2$, $g = G/\omega C_1$ where $\omega = 1/\sqrt{LC_2}$.

In spite of intensive theoretical and experimental studies, the question of principal bifurcations with which the birth of the double-scroll quasiattractor in the model is connected is not quite clear till now. The usual explanation based on the Shil'nikov theorem [1970] concerning the bifurcation of saddle-focus homoclinic loops is not completely satisfactory here because the hyperbolic set lying near the loop is not attractive. Therefore, the establishing of the presence of such loops is not sufficient for the existence of chaotic attractors. It will be shown, for instance, that for system (1) there exists a region in the parameter space where the bifurcational set corresponding to a single-round homoclinic loop of a saddle-focus lies, and Shil'nikov conditions are satisfied, but there is no chaotic attractor and most of orbits tend to a stable limit cycle.

In the present paper it is established that the appearance of the double-scroll is connected with the presence of heteroclinic orbits of intersection of two-dimensional invariant manifolds of the saddle-focus and a saddle periodic orbit. In one case this is one of periodic orbits lying in the Rossler attractor; it may also be a symmetric periodic orbit arising through a condensation of orbits. This assertion is based on the study of a model map by the use of which the birth and the structure of an attractive limit set can be described which is the intersection of the double-scroll with a cross-section.

Note that the double scroll contains the saddle-focus which may have homoclinic loops; also, the double-scroll may contain structurally unstable homoclinic orbits of saddle periodic orbits. By virtue of [Ovsyannikov & Shil'nikov, 1987; Gavrilov & Shilnikov, 1973; Newhouse, 1979], this implies that stable periodic orbits may appear in the double-scroll and it is, therefore, a quasiattractor [Shil'nikov, 1994]. This is the reason why the bifurcational set which corresponds to the birth of the double-scroll and which is a smooth curve contains a Cantor set of points of intersection with bifurcational curves corresponding to the situation where the one-dimensional separatrix of the saddle-focus belongs to the stable manifold of some nontrivial hyperbolic set. In the adjoint intervals the separatrix, apparently tends to one of the stable periodic orbits. A component of the bifurcational curve of the birth of the double-scroll can also be found which corresponds to the tangency of the stable and unstable manifolds of the hyperbolic set.

2. The Scenario of Transition to Chaos

2.1. *Local bifurcations*

Consider the sequence of the basic bifurcations with which the appearance of complex limit sets is connected. We begin with the study of equilibrium states. When $g > \alpha$ there exists only one equilibrium state O in the origin. When $g < \alpha$ there also exist two symmetric equilibrium states O_1, O_2 with the coordinates $x^*_{1,2} = \pm\sqrt{1 - g/a}$, $y^* = 0$, $z^*_{1,2} = \mp\sqrt{1 - g/\alpha}$. In this region of the parameter space O is unstable: it is a saddle or a saddle-focus with one positive characteristic exponent. The one-dimensional separatrices of O will be denoted as P_i, $i = 1, 2$, and the two-dimensional stable manifold of O will be denoted as $W^s(O)$.

On the line $g = \alpha$ the characteristic equation of 0 has one zero root for $\beta \neq 1/\alpha^2$ and it has two zero roots at $\beta = 1/\alpha^2$. We denote this point on the parameter plane $(\beta, g/\alpha)$ as TH (Takens–Harozov). The bifurcations in the neighborhood of this point are known [Harozov, 1979] to be determined by the following normal form on the center manifold:

$$\dot{x} = y, \quad \dot{y} = \varepsilon x + \mu y + ax^3 + \gamma x^2 y\,.$$

In our case, for system (1) we have $a = -1/\alpha^2$, $\gamma = 3(\alpha^2 - 1)/\alpha^3$, and the bifurcation diagram corresponding to $\alpha < 1$ has the form shown in Fig. 3 [Harozov, 1979 & Arnold, 1982]. Note the presence of the curve SL which corresponds to a homoclinic loop of the saddle O. In a sufficiently small

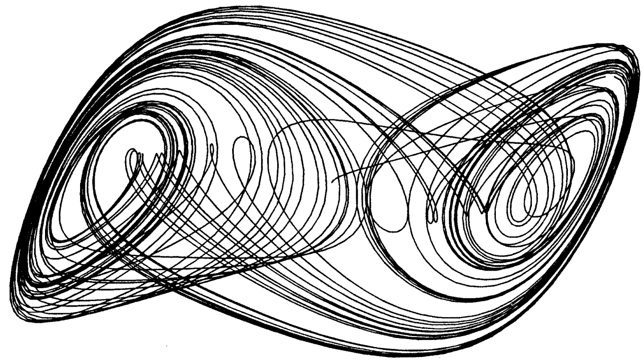


Fig. 2. The double-scroll quasiattractor.

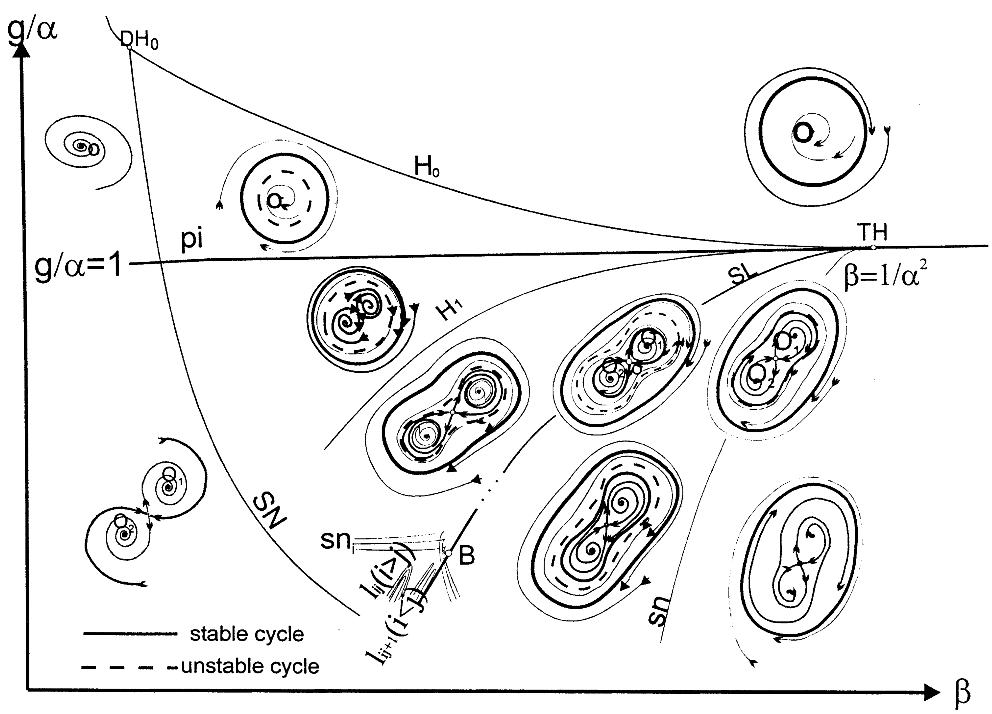


Fig. 3. The local bifurcation diagram on the β–g/a plane at $a = 0.2$. Three codimension-2 bifurcation points (TH, Takens–Harozov; DH_0, a degenerate Andronov–Hopf bifurcation of the origin; B, Bel'jakov point — a homoclinic loop to a saddle-focus with the saddle index ν equal to 1) and several codimension-1 bifurcation curves (PI, pitchfork of equilibria; H_0, Andronov–Hopf of the origin; H_1, Andronov–Hopf of nontrivial equilibria; SN and sn, saddle-node bifurcation of periodic orbits; SL — homoclinic loop of the origin) are present.

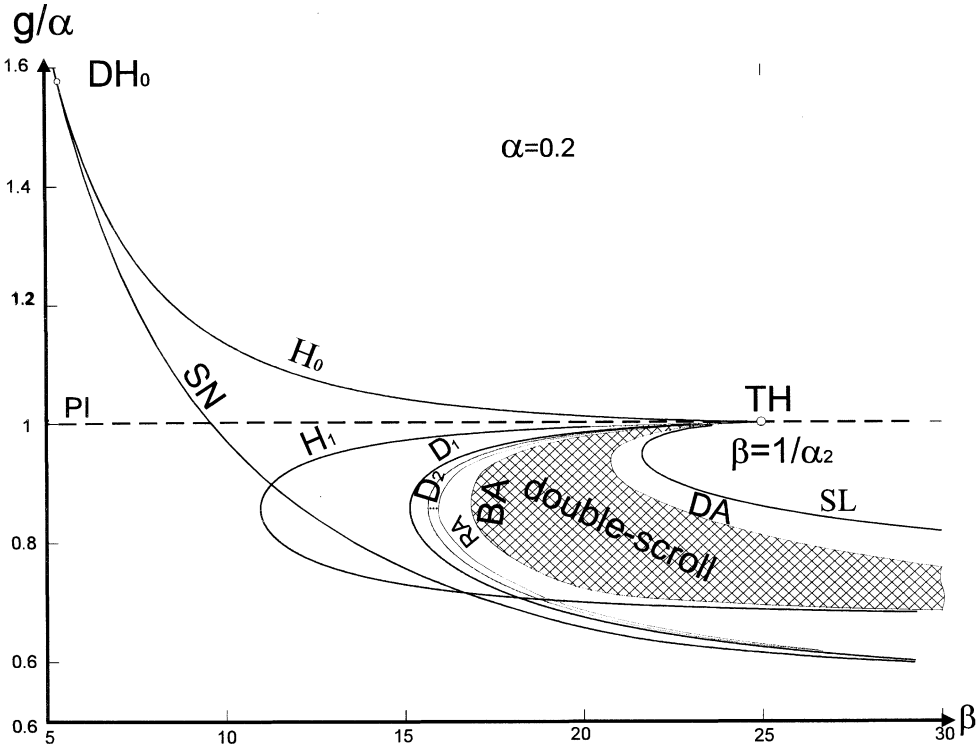


Fig. 4. The global bifurcation diagram on β–g/a plane at $a = 0.2$. The points TH, DH_0 and the curves H_0, H_1, SN, sn and SL are the same as in Fig. 3. The curve D_1 is the curve of a period-doubling, D_2 is the curve of secondary period-doubling, Ra is the curve of the birth of Rossler attractor as a result of an infinite sequence of period-doublings, Ba is the curve of the birth of the double-scroll quasiattractor, and Da is the curve of the death of the double-scroll.

neighborhood of the point TH the saddle value is negative in O, so a stable limit cycle is born from the loop. This cycle coalesces with an unstable cycle on the curve sn which corresponds to the saddle-node periodic orbit. All bifurcational curves starting with the point TH are continued outside its small neighborhood.

2.2. *The bifurcation diagram*

Let us now consider nonlocal bifurcations. A calculation of the first Lyapunov value $L(\beta)$ on the curve H_0 shows that there exists a point DH_0 where the Lyapunov value vanishes. On the segment $TH - DH_0$ we have $L(\beta) > 0$, and $L(\beta) < 0$ above the point DH_0. A bifurcation curve SN that corresponds to a symmetric periodic orbit with one multiplier equal to $+1$ goes from the point DH_0. To the right of this curve there exist two symmetric periodic orbits: the stable periodic orbit Γ_0 and the saddle periodic orbit L_0. When moving on the parameter plane near the point TH, the latter disappears on the curve SL (in the region $g < \alpha$).

The bifurcation diagram is shown in Fig. 4. The main phenomena mentioned above are connected with the curve SL. There exists a bifurcation point B on SL which corresponds to the saddle exponent $\nu = -\mathrm{Re}\,\lambda_{1,2}/\lambda_3$ equal to unity where λ, $\lambda_{1,2}$ are the characteristic exponents of the saddle-focus 0. According to [Belyakov, 1984], below the point B there exists a nontrivial hyperbolic set in the neighborhood of the homoclinic loop of the saddle-focus. It follows from [Belyakov, 1984] that below the point B there exist infinitely many bifurcational curves l_{ij} and l_{ij+1} corresponding to multi-round homoclinic loops and infinitely many bifurcational curves sn_i corresponding to saddle-node periodic orbits.

Different types of bifurcation sequences leading to chaotic dynamics were described in [Shil'nikov, 1991]. In our case one of the typical scenarios can be: equilibrium state–periodic orbit–period doubling–chaos, as well as the scenario connected with structurally unstable Poincaré homoclinic orbits arising before the period-doubling cascade finishes (see, for instance [Dmitriev *et al.*, 1992]).

We describe here (following Shil'nikov [1991]) the main features of the first scenario. We will trace the periodic orbit P_i which is born from O_i. Note that P_i is the boundary of the unstable manifold

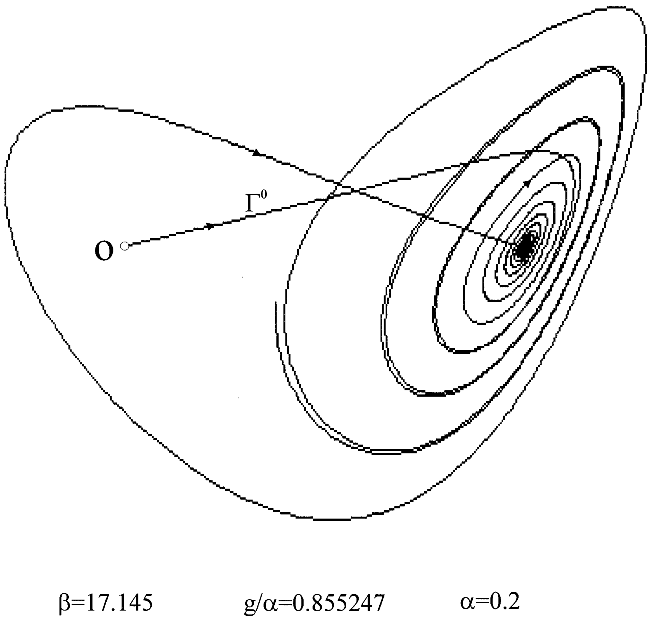


Fig. 5. The separatrix contour which contains the two saddle-foci.

$W^u(O_i)$; it is also the limit set for a separatrix of the saddle-focus O. Later (when the parameter β increases) the multipliers of this periodic orbit become complex-conjugate which leads to the situation where the manifold $W^u(O_i)$ becomes winding onto P_i. Next, on the curve D_1, the period-doubling bifurcation occurs with the orbit Γ_i and then the period-doubling cascade starts. This process leads to the appearance of two nonsymmetric Rossler attractors $\mathcal{M}_1$ and $\mathcal{M}_2$ which inherit to P_1 and P_2.

In the parameter region where the quasiattractor $\mathcal{M}_i$ is separated from the other one by the stable manifold $W^s(0)$, there is a limit set for $W^u(O_i)$ as well as for the one-dimensional separatrix Γ_i. The beginning of the birth of a symmetric quasiattractor is the tangency of $W^u(O_1)$ with $W^s(O)$ along a heteroclinic orbit (due to the symmetry there also appears a heteroclinic orbit of tangency of $W^u(O_2)$ with $W^s(O)$). When the parameter varies, each of these heteroclinic orbits split onto two transverse heteroclinic curves. In this situation both one-dimensional separatrices Γ_1 and Γ_2 of the saddle-focus O belong simultaneously to the boundary of each of the manifolds $W^u(O_i)$.

In the region of the existence of a double-scroll there exists a bifurcation point Q ($\beta \approx 17.145$, $g \approx 0.855247$) of the codimension two which corresponds to the existence of a separatrix contour L (Fig. 5). This contour is formed by a trajectory Γ^0 (which is formed by coinciding one-dimensional separatrixes Γ_0 and Γ_1 of saddle-foci O and O_1 respectively) and by a rough heteroclinic trajectory. This heteroclinic trajectory belongs to an intersection of two-dimensional manifolds of the saddle-foci. This point corresponds to the existence of a nontrivial hyperbolic set, in the neighborhood of the contour, and a countable set of heteroclinic trajectories belonging to an intersection of two-dimensional manifolds of the saddle-foci. Under the conditions

$$G \equiv \nu_1^2/\omega_1^2 + \nu_2^2/\omega_2^2 + 2\nu_1\nu_2/(\omega_1\omega_2)\varepsilon + 1 - \varepsilon^2 > 0\,,$$

where $\varepsilon > 1$ ($\varepsilon > 1$) is some value which is determined via solutions of equations in variations when integrating along the trajectory Γ^0, $\nu = -\mathrm{Re}\,\lambda_{1,2}/\lambda_3$ is the saddle index, $\lambda_{1,2}$, λ_3 are complex conjugated and real roots in the saddle-focus respectively, and ω_i is the imaginary part of eigenvalues of saddle-foci. On the bifurcation set there are everywhere dense systems belonging to a nonrough heteroclinic trajectory on which there takes place a tangency of the two-dimensional manifolds of the saddle-foci. Besides, the point Q is a limiting point for a countable set of other bifurcation points Q_i which correspond to the existence of separatrix contours having properties as that of L.

3. The Model Map

In this section we propose a geometric model which describes the main features of system (1). On the other hand our construction is interesting itself and may be considered as a realization of the object called "confinor" in [Lozi & Ushiki, 1991].

Suppose that smooth dissipative system X possessing a center symmetry has a saddle-focus equilibrium state O with the characteristic exponents $\lambda_{1,2} = -\lambda \pm i\omega$, λ_3 where λ, ω and λ_3 are positive. Also, the system is supposed to have a symmetric saddle periodic orbit L_0. Assume that the stable manifold $W^s(L_0)$ of the periodic orbit is homeomorphic to a cylinder and the saddle-focus O lies inside the cylinder. We also suppose that outside $W^s(L_0)$ there is a stable symmetric periodic orbit Γ_0, like in system (1).

Let D be some cross-section on which Cartesian coordinates (x, y) can be introduced so that the line $l_0 : y = 0$ is the intersection with the manifold $W^s(O)$. Denote as p_i the point of the first intersection of one-dimensional separatrix Γ_i of O with D. Let $U(l_0)$ and $U_i(p_i)$ be some neighborhoods of the line l_0 and of the point p_1

respectively and let $U^+(l_0)$ ($U^-(l_0)$) be the component of $U(l_0)$ corresponding to the positive (respectively negative) values of y. According to [Gavrilov & Shilnikov, 1973; Shil'nikov, 1994], the map $U^+(l_0) \to U_1(p_1)$ defined by the orbits of the system is represented in the form

$$\bar{x} = x^* + b_1 \cdot x \cdot y^\nu \cos(\omega \cdot \ln(y) + \theta_1) + \psi_1(x, y)$$
$$\bar{y} = y^* + b_2 \cdot x \cdot y^\nu \sin(\omega \cdot \ln(y) + \theta_2) + \psi_2(x, y) \qquad (2)$$

where ψ_i are smooth functions which tend, as $y \to 0$, to zero along with their first derivative with respect to y. An analogous formula is valid for the map $U^-(l_0) \to U_2(p_2)$. An extrapolation of the properties of the local map defined by (2) leads to the following construction which is in a good agreement with the results of computer simulation.

Consider two components D^+ and D^- into which the cross-section D is divided by the line l_0; i.e. $D = D^+ \bigcup D^- \bigcup l_0$. We have $D^+ = \{(x, y)||x| \le c, 0 < y \le h^*(x)|\}$, $D^- = \{(x, y)||x| \le c, -h(x) \le y < 0\}$, where $y = h^*(x)$ is a component of the intersection of the stable manifold $W^s(L_0)$ with D. The orbits of the system define the maps $T(\mu)^+ : D^+ \to D$, $T(\mu)^- : D^- \to D$, and $T(\mu)^+$ and $T(\mu)^-$ are written as

$$\bar{x} = f^+(x, y, \mu), \quad \bar{y} = g^+(x, y, \mu)$$

and

$$\bar{x} = f^-(x, y, \mu), \quad \bar{y} = g^-(x, y, \mu)$$

respectively; here, $f^+, f^-, g^+, g^- \in C^r$, and $\mu = (\mu^{(1)}, \mu^{(2)})$.[1]

The following properties are assumed for these maps.

(1) T^+ and T^- can be defined on l_0 so that $\lim_{y \to +0} T^+M(x,y) = (x^*, y^*)$, $\lim_{y\to -0} T^-M(x, y) = (x^{**}, -y^*)$, where $p_1(x^*, y^*)$ and $p_2(x^{**}, -y^*)$ are the points of the first intersection of one-dimensional separatrices of the saddle-focus O with D. We suppose that T depends on μ in the following way: the first component of μ shifts the point P_1 in vertical direction and the second component of μ shifts it in horizontal direction.

(2) Each of the regions D^+, D^- are represented as a union of an infinite number of regions $\tilde{S}_0^+$, $\tilde{S}_0^-$, S_i^+, S_i^- $S_i^+ = \{(x, y)||x| \le c, \xi_{i+1}^* \le y < \xi_i^*\}$, $S_i^- = \{(x, y)||x| \le c, -\xi_i^* < y \le -\xi_{i+1}^*\}$, $i = 1, \ldots,$ where $\xi_0^* = h^*(x)$, $\lim_{i\to\infty} |\xi_i^*| = 0$. The map T^+ or T^- acts so that the image of any vertical segment with one end-point on l_0 has the form of a spiral winding at the point p_1 or p_2, respectively. The boundary $\gamma_i^\pm : y = \xi_{i+1}^*$ of two adjoining regions $S_{i,}^\pm S_{i+1}^\pm$ is chosen so that $\partial\bar{y}/\partial y = 0$ if and only if $(x, y) \in \gamma_i^\pm$, and $T^\pm\gamma^\pm$ is a segment of a curve of the form $x = h_i(y)$ where $|dh/dy| < 1$.

Introduce the following notations:

$$D_1 = \{(x, y)| -c \le x \le -c/2,\ |y| < h(x)\},$$
$$D_2 = \{(x, y)|c/2 \le x \le c,\ |y| < h(x)\},$$
$$D_i^+ = D^+ \bigcap D_i,\ D_i^- = D^- \bigcap D_I,$$
$$T \equiv T^\pm|D^\pm,\ (f^\pm, g^\pm) \equiv (f, g),$$
$$S_i \equiv S_i^\pm,\ \gamma_i \equiv \gamma_i^\pm$$

Let S_0^+ (S_0^-) be that part of $\tilde{S}_0^+$ ($\tilde{S}_0^-$) on which $TS_0^+ \in D_1$ ($TS_0^- \in D_2$). Suppose that the map T satisfies the following additional conditions:

(3) $TD_1 \subset D$, $TD_2 \subset D$;
(4) $|\partial\bar{x}/\partial x| < 1$;
(5) $r_i = \varrho(T\gamma_i, T\gamma_{i+1}) > q\xi_i^*$, $q > 2$, $i > 1$;
(6) in S_i there can be a set σ_i selected such that the following inequality is fulfilled everywhere on σ_i:

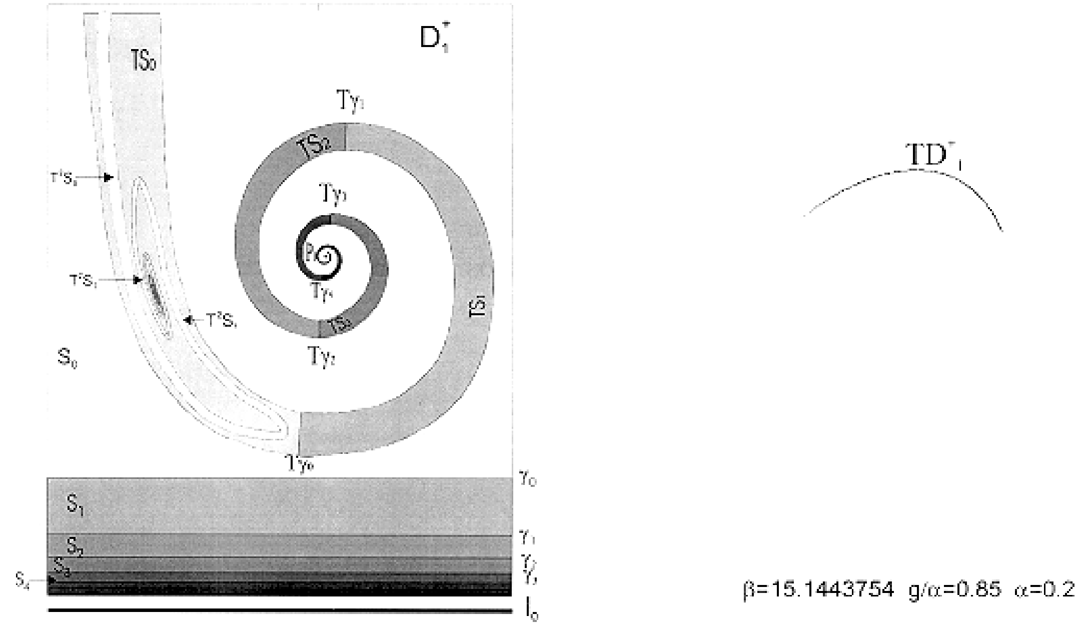
$$\|g_y\| > 1,$$
$$1 - \|f_x \cdot g_y^{-1}\| < 2\sqrt{\|f_y \cdot g_y^{-1}\| \cdot \|g_x \cdot g_y^{-1}\|}$$

(7) $p_1 \in S_0^+$, $p_2 \in S_0^-$

Note that condition (2) is fulfilled near the line l_0 (i.e. for the sets S_i with i sufficiently large) by virtue of (2). Conditions (3)–(5) are also automatically fulfilled near l_0 if the saddle index $\rho = -\mathrm{Re}\,\lambda_{1,2}/\lambda_3$ is less than unity in the saddle-focus. Moreover, in this case $r_i/\xi_i^* \to \infty$ as $i \to \infty$; in other words, $q \to \infty$ as $i \to \infty$.[2]

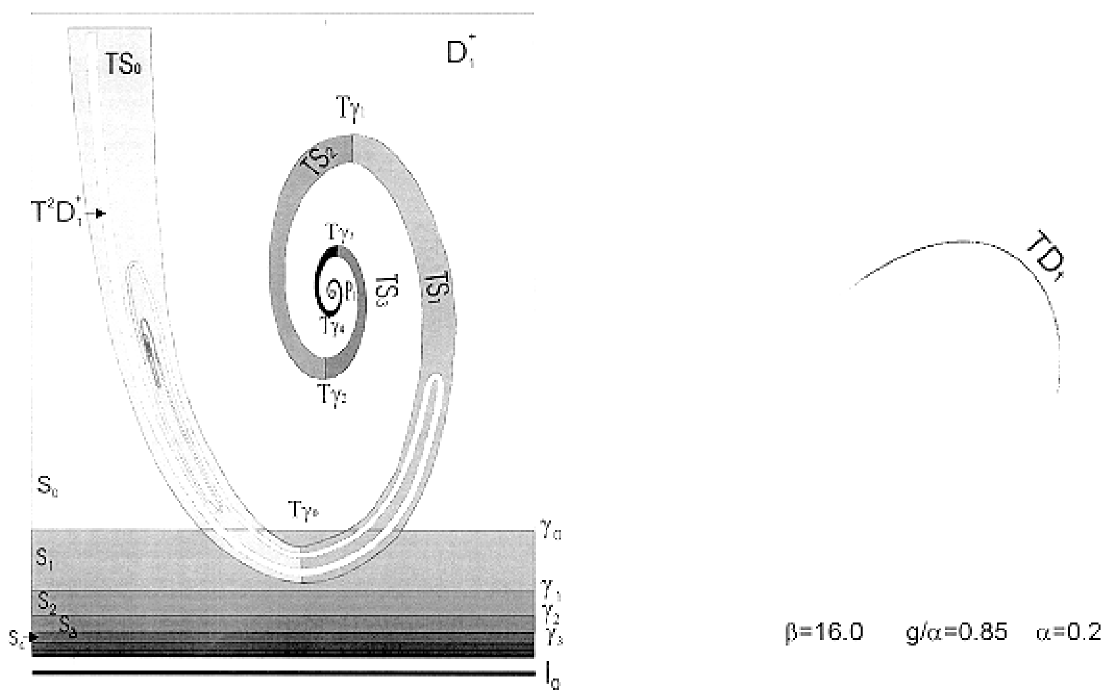
[1]The original double-scroll model map of the Chua attractor, see [Belykh & Chua, 1992].

[2]The validity of such a model can be verified by computer simulations. It occurs that the contraction in horizontal direction is so strong that the images of the region $x = \sqrt{1 - g/a}/2$ under the action of the Poincaré map have, in natural scale, the form of one-dimensional curves [Figs. 6(b)–9(b) and 13(b)]



(a) (b)

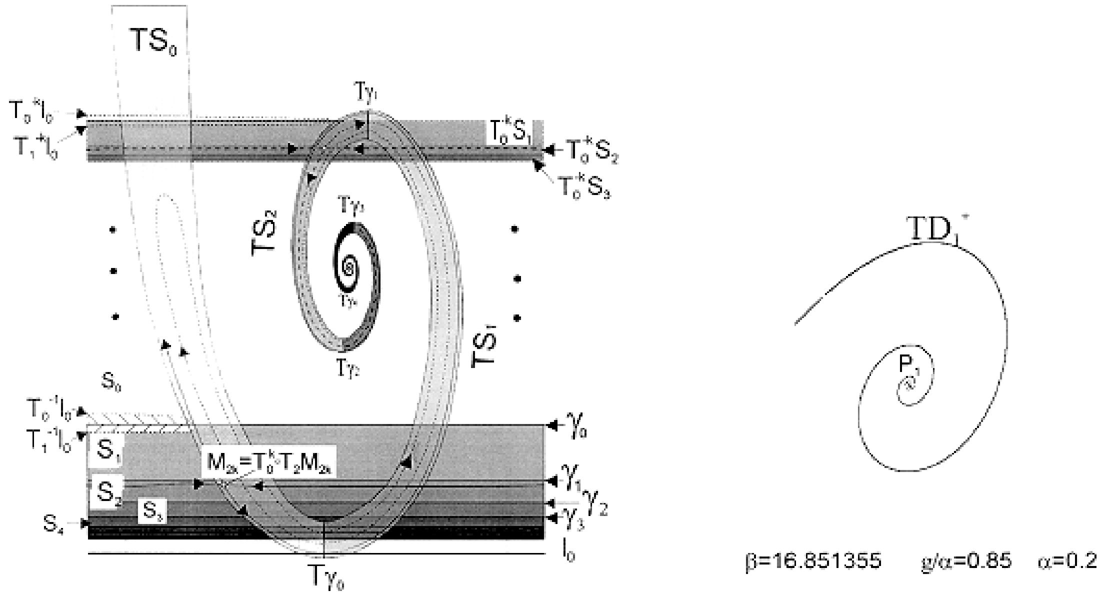
Fig. 6. The Poincaré map on the semi-plane $x = \sqrt{(1-g/a)}$, $y > \sqrt{(1-g/a)}(\beta - 1/4)/2\beta$ for the parameter values lying to the left of the bifurcation curve D_1; (a) the theoretical model, (b) computer simulations.



(a) (b)

Fig. 7. As Fig. 6 with values of β and g/a lying to the right of the curve D_1.

The fulfillment of condition (6) means that for those parameter values for which there exist i and j such that the map $\sigma_i \to \sigma_j$ is defined, the operator $\mathcal{T}_{ij} : H_i(L) \to H_j(L)$ is contracting [Shashkov & Shil'nikov, 1994; Afraimovich & Shil'nikov, 1973], where $H_i(L)$ is the space of the curves $y = \varphi(x)$ lying in σ_i and satisfying the Lipschitz condition with some Lipschitz constant L.



(a) (b)

Fig. 8. As Fig. 6 with values of β and g/a lying on the curve BA.

For system (1), one can see that the divergence $\mathrm{Div} = \beta(\alpha - g) - g$ of the vector field in the saddle-focus O is positive for $g < \beta\alpha/(1+\beta)$. This implies that $2\nu < 1$ in this parameter region. For us, it is important that $\nu < 1$ which, for the case of formation of a homoclinic loop, implies the presence of nontrivial hyperbolic sets and infinitely many Smale horseshoes.

Let T_k be the restriction of T onto S_k. For those parameter values for which $T\gamma_0 \subset S_0$, $T\gamma_0 \bigcap S_1 = \emptyset$ (i.e. $T_0 S_0 \subset S_0$ as in Fig. 6), in S_0 there exists a stable limit set which depends on the concrete properties of the map T_0. When the parameter changes so that the image of the boundary γ_0 moves down, from the moment when $T\gamma_0 \bigcap S_1 \neq \emptyset$, the situation becomes analogous to the creation of the Smale horseshoe (Fig. 7). In addition to the fixed point $M_1 = T_1 M_1$, saddle periodic points $M_{1k} = T_0^k \circ T_1 M_{1k}$, $1 < k \geq k^+$ arise in S_1 and S_0. The appearance of these points implies the creation of a nontrivial hyperbolic set due to the formation of a heteroclinic contour composed by heteroclinic orbits of intersection of the stable (unstable) manifolds of each of these points with the unstable (stable) manifolds of each other point.

The further shift downwards of the curve $T\gamma_0$ implies the appearance of periodic points $M_{2n} = T_0^n \circ T_2 M_{2n} n > k_2^-$ in S_0 and S_2, and so on. The structure of the limit set becomes more and more complicated but it remains in the region D_1^+ for the moment when the unstable manifold of the point M_{2n} has a tangency with l_0 (Fig. 8). After the moment of tangency of $T_0 S_0 \cap l_0 \neq \emptyset$ and $T_1 S_1 \cap l_0 \neq \emptyset$ there appear preimages $T_0^{-1} l_0$ and $T_1^{-1} l_0$ of the line l_0 in S_0 and S_1 near the boundary γ_0. These preimages intersect the unstable manifold of the nontrivial hyperbolic set. Thus, the preimages of l_0 will accumulate on the leaves of the stable manifold of the hyperbolic set. Besides, those parts of S_0^+ and S_1^+ which are bounded by the curves $T_0^{-1} l_0$ and $T_1^{-1} l_0$ are mapped by T^+ into D^- (i.e. in the region where the map T^- acts). Due to the symmetry, the analogous parts of S_0^- and S_1^- are mapped into D^+ by T^-. Thus, two symmetric attractors are now united. The limit set newly created is the double-scroll attractor which we will consider in the next section.

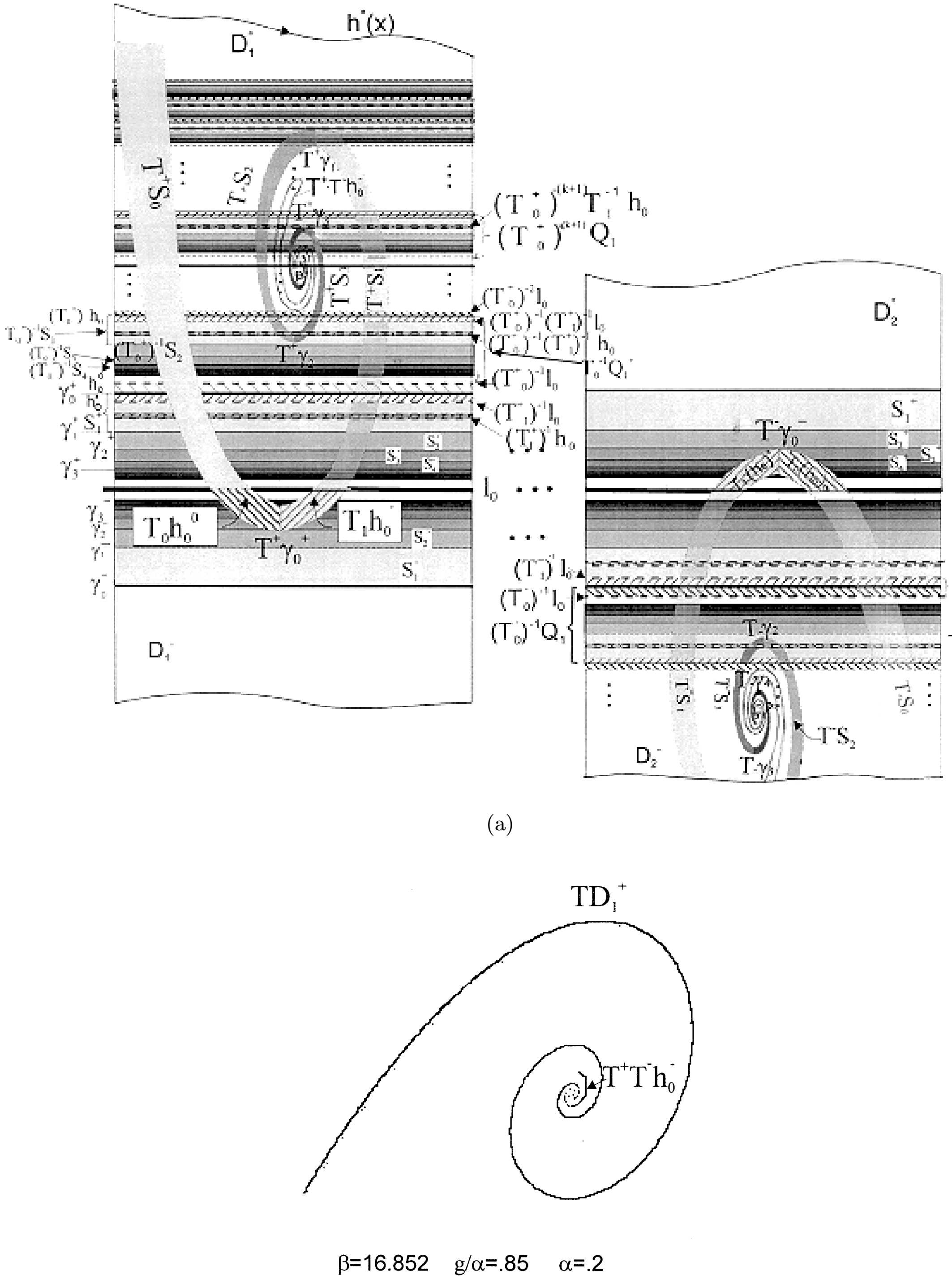
4. The Structure of the Limit Set

Denote the region bounded by the lines $T_0^{-1} l_0$ and $T_1^{-1} l_0$ as h_0. Consider the regions

(a) h_0^0 and h_0^1 which are the connected components of $h_0 \backslash \gamma_0$;

(b) $Q_1 = \bigcup_{i=1}^{\infty} S_i$

Let $\bar{k}$ be the number such that the point p_1



(b)

Fig. 9. As Fig. 5 with values β, g/a lying to the right of the curve BA.

belongs to $T_0^{-\bar{k}}(Q_1 \bigcup h_0^0)$. Denote

$$k_1^+ = \max\{k | T_0^{-k}(Q_1 \bigcup h_0^0) \cap T_1 S_1 \neq \emptyset\,,$$
$$T\gamma_1 \cap T_0^{-k}(Q_1 \cup h_0^0) = \emptyset\}$$

and

$$R_1 = S_1 \Bigg\backslash \left(h_0^1 \bigcup_{k=1}^{k_1^+} (T_1^{-1} \circ T_0^{-k}(Q_1 \bigcup h_0^0)) \right) .$$

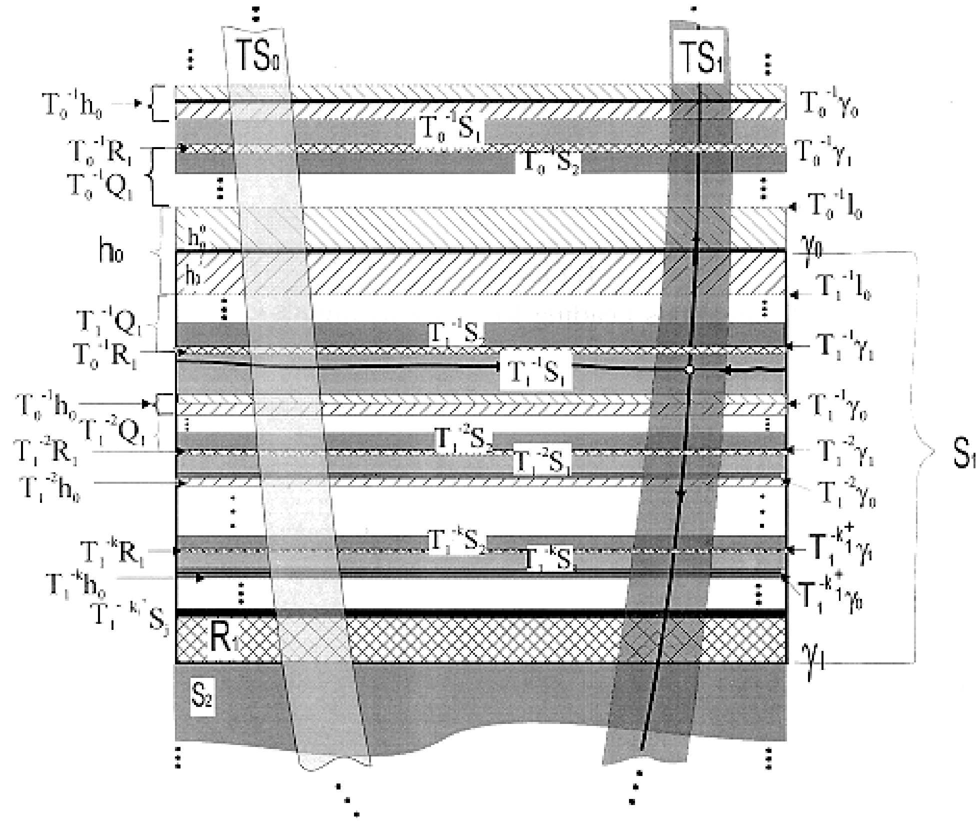


Fig. 10. The decomposition of the region S_1 onto the regions corresponding to different behavior of orbits (an illustration to Lemma 3.1).

By these definitions, $k_1^+ > \bar{k}$ is the maximal number such that $T_1\gamma_1 \bigcap T_0^{-k_1^+}(Q_1 \bigcup h_0) = \emptyset$ and $T_1\gamma_1 \bigcap T_0^{-k_1^+-1}(Q_1 \bigcup h_0) \neq \emptyset$; and R_1 is the complement to the preimage of the regions $T_0^{-k)}(Q_1 \cup h_0^0)$ with respect to the map T_1.

Let $X \subset D_1^+$ be a region bounded by two vertical segments lying on the boundaries of D_1^+ and by two horizontal lines lying between $T\gamma_0$ and $T\gamma_1$. Let X_0 and X_1 be the intersections $TS_0 \cap X$ and $TS_1 \cap X$, respectively. If $T\gamma_i \cap X_j = \emptyset$, $i, j = 0, 1$, then the left and right boundaries of these sets will again lie on the corresponding boundaries of D_1^+.

We define inductively the subsets of D_1 so that if a point belongs to such subset, this will determine its behavior under the action of the map T.

(1) $X_i^0 = X_i$, $i = 0, 1$

(2) $X_i^l = T_i^{-1}(X_i^{l-1} \bigcap TS_i)$

Evidently, $X_i^{l_1}$ is the l_1th preimage of X_i with respect to T_i. Analogously to (1), we denote $X_i^{l_1} \cap TS_j = (X_i^{l_1})_j^1$. Applying, $l_2 - 1$ times, the map T_j^{-1} we obtain, analogously to (2), the set $(X_i^{l_1})_j^{l_2}$ or, omitting parentheses, $X_{i,j}^{l_1,l_2}$. Thus

(3) $X_{n_1,n_l,\ldots,n_{l-1},n_l}^{m_1,m_2,\ldots,m_{l-1},1} = T_{n_l}^{-1}(X_{n_1,n_2,\ldots,n_{l_1-1}}^{m_1,m_2,\ldots,m_{l-1}} \bigcap TS_{n_l})$, where $n_i = 0, 1$; $0 \leq m_i \leq k_1^+$ if $n_i = 0$, and $m_i \in \mathbf{z}$ if $n_i = 1$.

The set of regions defined by (3) is in one-to-one correspondence with the set of infinite (to the left) sequences of zeros and units with the restriction that the length of complete strings of zeros must not be greater than k_1^+ [because $m_j \leq k_1^+$ if $n_j = 0$; see (3)].

As the region X, one can, for instance, take the region S_1 because $S_1 \cap TS_0 \neq \emptyset$ and $S_1 \cap T_1S_1 \neq \emptyset$ in the situation where $T\gamma_0$ lies below l_0. Following the above scheme (1)–(3) the set $\{(S_1)_{\mathbf{m_s}}^{\mathbf{n_s}}\}$ of regions $(S_1)_{\mathbf{m_s}}^{\mathbf{n_s}}$ is constructed where $\mathbf{m_s} = (m_1, m_2, \ldots, m_{i_s})$, $\mathbf{n_s} = (n_1, n_2, \ldots, n_s)$ are multi-indices. The limit $s \to \infty$ corresponds to a set S_1^* which is in one-to-one correspondence with the set of sequences infinite to the left, composed by zeros and units with the restriction that the length of any complete string of zeros does not exceed k_1^+.

The set S_1^* consists of invariant fibers; it can also be shown by the use of condition (6) that the set S_1^* contains a nontrivial hyperbolic set: each fiber contains exactly one point of this set. Note that the nonwandering set is not, in principle, exhausted by the orbits of S_1^*.

The following lemma describing the structure of the decomposition of S_1 onto regions corresponding to different types of orbit behavior is evident (Fig. 10).

Lemma 3.1. *The region S_1 is a union of the following sets*:

(a) S_1^* *is the set of stable fibers*; *the set of points whose orbits never leave $S_0 \bigcup S_1$ under the action of the map T*;
(b) $H^* = h_0^1 \bigcup (\bigcup_{s\geq 1}^{\infty}(h_0)_{\mathbf{n}_s}^{\mathbf{m}_s} \cap S_1$ *is the set of points whose orbits leave D_1^+ and enter D_1^- after a finite number of iterations*;
(c) $R_1^* = \bigcup (R_1)_{\mathbf{n}_s}^{\mathbf{m}_s}) \cap S_1$ *by the set of points whose orbits enter R_1 after a finite number of iterations*; *this set may contain stable periodic orbits*;
(d) $Q_1^* = \bigcup_{s\geq 1}^{\infty}(Q_1 \backslash S_1)_{\mathbf{n}_s}^{\mathbf{m}_s} \cap S_1$, *is the set of points whose orbits enter one of the regions S_i, $1 < i < \infty$ after a finite number of iterations*;
(e) $L_1^* = \bigcup_{s\geq 1}^{\infty}(l_0)_{\mathbf{n}_s}^{\mathbf{m}_s} \cap S_1$, *is the set of preimages of the discontinuity line l_0 where $\mathbf{m}_s = (m_1, m_2, \ldots, m_{j_s})$, $\mathbf{n}_s = (n_1, n_2, \ldots, n_s)$, $m_i \in \{0, 1\}$, $m_{j_s} = 1$; $1 \leq n_i \leq k_1^+$ if $m_i = 0$ and $n_i \in \mathbf{Z}$, if $m_i = 1$.*

We note that there exists such a k, for which $T_0^{-k}(S_1 \cup S_2) \cap T_2 S_2 \neq \emptyset$, and $T_0^{-k}(S_1 \cup S_2) \cap T(\gamma_1 \cup \gamma_2) = \emptyset$. Then S_2 will contain the preimage $T_2^{-1} \circ T_0^{-k} S_1$, and all preimages of the regions S_i, $i = 1, 2, \ldots, h_0$ and $(R_1)_1^*$. One can point out the same properties for other regions S_i, i.e. there exist such $\bar{m}$, j_i^* and k_i^-, k_i^+, $1 \leq k_i^- \leq k_i^+$, where $1 < i \leq \bar{m}$, $1 \leq j_i^* \leq i-1$, that for $k_i^- \leq k \leq k_i^+$ as $1 < i \leq \bar{m}$ the following situation occurs: $T_0^{-k} S_j \cap T_i S_i \neq \emptyset$, $T_0^{-k}(\bigcup_{j_1^* \leq i} S_j) \bigcap T(\gamma_i \cup \gamma_{i+1}) = \emptyset$. Hence, in the same manner as in the case of the region S_i, one can present the decomposition S_m, $m = 2, 3, \ldots, \bar{m}$ in preimages S_i, h_0, S_i/σ_i, $i = 1, 2, \ldots, \bar{m}$ and $Q_m = \bigcup_{i=m}^{\infty} Q_i$. Then an arbitrary region S_i, $1 \leq i \leq \bar{m}$ will contain preimages of the regions S_j, $j \leq i$.

In order to describe the set of preimages of the line l_0 and of the regions S_i let us construct the graph G, in which, following [Afraimovich & Shil'nikov, 1973], edges denote states, while vertices denote transformations. Let each region S_i correspond to a vertex a_i, while the saddle-focus O correspond to an edge $\hat{O}$, whose beginning and end form the vertex 0; the maps $T_j^{-1} \circ T_0^{-k} S_i$: $S_i \to S_j$ correspond to edges $\hat{b}_{ij}^k$, if these maps are defined and $T(\gamma_j \cup \gamma_{j+1}) \cap T_0^{-k} S_i = \emptyset$ and, at last, the trajectories, which are asymptotic to O as $t \to \infty$ and start on D at points belonging to $T_1^{-1} l_0$ and $T_0^{-k} l_0 \bigcap S_i$, correspond to edges $\hat{1}$ and $\hat{2}$ respectively, which come out from the vertex 0 and end at the vertex a_1, if $T_0^{-k} l_0 \bigcap S_1 \neq \emptyset$ and $T_0^{-k} l_0 \bigcap T(\gamma_i \bigcup \gamma_{i+1}) = \emptyset$. Let us leave only those vertices which have at least one incoming and outgoing edges, while we remove the rest of the vertices together with their starting edges. Consider the graph G and its subset G_σ, which contains only those edges $\hat{b}_{ij}^k$ for which $T_j^{-1} \circ T_0^{-k} S_i \subset \sigma_j$.

Since, according to the construction, each of the edges $\hat{b}_{ij}^k$ of the graph G_σ are determined by the map $T_j^{-1} \circ T_0^{-k} \sigma_i \to \sigma_j$, and regions σ_i are distinguished by condition (6), then, in the same manner as in [Afraimovich *et al.*, 1983; Shashkov & Shil'nikov, 1994] one can show that the map $T_j^{-1} \circ T_0^{-k}$ generates the operator $\mathcal{T}_{ij}$: $H_i(L) \to H_j(L)$ satisfying the squeezed-map principle. Since the number of edges in the graph G_σ does not exceed a countable set, we numerate them so that each edge will correspond to a natural number. Let Ω denote the space of all sequences of the form: $(\ldots, \kappa_{i-1}, \kappa_i, \kappa_{i+1}, \ldots)$, where the symbol κ_i corresponds to the edges of the graph G_σ in such a way that two nearest symbols κ_{i-1}, κ_i denote edges, one of which ends and the other starts at their common vertex. Then each point $\omega = (\ldots, \kappa_{i_{-1}}, \kappa_{i_0}\kappa_{i_1}, \ldots)$ in Ω is in one-to-one correspondence with a sequence of spaces and maps

$$\ldots,\ H_{i_{-1}} \overset{\mathcal{T}_{i_{-1}i_0}}{\longrightarrow} H_{i_0} \overset{\mathcal{T}_{i_0 i_1}}{\to} H_{i_1}, \ldots.$$

Since all the spaces are complete and all operators are squeezing, then according to the lemma from [Shil'nikov, 1968] there exists a unique sequence of curves $(\ldots, h_{i_{-1}}, h_{i_0}, h_{i_1}, \ldots)$. This sequence we shall call the invariant stable fibre.

As follows from Lemma 3.1, the sets $Q_1^* \bigcup H_1^* \bigcup R_1^*$ give the adjoint intervals in S_1 for the Cantor discontinuum of the set S_1^* of stable fibers of the nontrivial hyperbolic set. Evidently, if $p_1 \in H_1^*$, then after some number of iterations the point p_1 will lie below the line l_0 and the next part of its orbit will be defined by the map T^-. If $p_1 \in S_1^*$ (i.e. if it belongs to some stable fiber), then there exist arbitrarily small perturbations which move the point p_1 into H_1^*. Here, the point p_1 will cross preimages of the discontinuity line l_0. Thus, after a number of iterations, the point p_1 may be mapped close to the line l_0, at a distance which is less than perturbations acting on the system, and the behavior of the following iterations of p_1 will not be defined uniquely.

For the set S_2 the number k can also be found such that $T_0^{-k}(S_1 \cup S_2) \cap T_2 S_2 \neq \emptyset$ and $T_0^{-k}(S_1 \cup S_2) \cap T(\gamma_1 \cup \gamma_2) = \emptyset$. In this case S_2 will contain the preimage $T_2^{-1} \circ T_0^{-k} S_1$, together with the preimages of the regions S_i, $i = 1, 2, \ldots, h_0$ and $(R_1)_1^*$ defined by Lemma 3.1.

The same may hold for the other regions S_i; i.e. there exist $\bar{m}$, j_i^* and k_i^-, k_i^+ $(1 \leq k_i^- \leq k_i^+, 1 < i \leq \bar{m}, 1 \leq j_i^* \leq i-1)$ such that the following situation takes place for $k_i^- \leq k \leq k_i^+$ and $1 < i \leq \bar{m}$: $T_0^{-k} S_j \cap T_i S_i \neq \emptyset$, $T_0^{-k}(\bigcup_{j_1^* \leq i} S_j) \bigcap T(\gamma_i \cup \gamma_{i+1}) = \emptyset$. As in Lemma 3.1, the decomposition of the region S_i onto the sets S_i, h_0, S_i/σ_i, $i = 1, 2, \ldots, \bar{m}$ and $Q_m = \bigcup_{i=m}^{\infty} Q_i$ can be done. Note that any region S_i, $1 \leq i \leq \bar{m}$ will contain preimages of the regions S_j, $j \leq i$ (Figs. 7–9).

To describe the set of preimages of the discontinuity line l_0 and the regions S_i we consider the graph G defined by the following way. Each region S_i is represented by the vertex a_i; the saddle-focus O is represented by the vertex 0 with the edge $\hat{O}$ which starts and ends in 0; the maps $T_j^{-1} \circ T_0^{-k} S_i$: $S_i \to S_j$ are represented by the edges $\hat{b}_{ij}^k$ if these maps are defined and if $T(\gamma_j \cup \gamma_{j+1}) \cap T_0^{-k} S_i = \emptyset$; if $T_0^{-k} l_0 \bigcap S_1 \neq \emptyset$ and $T_0^{-k} l_0 \bigcap T(\gamma_i \bigcup \gamma_{i+1}) = \emptyset$, then the edges $\hat{1}$ and $\hat{2}$ are also constructed (Fig. 11).

Simultaneously, we consider the graph G_σ which can be obtained if to retain only those edges $\hat{b}_{ij}^k$ of G for which $T_j^{-1} \circ T_0^{-k} S_i \subset \sigma_j$. We also retain only those vertices for which there exist at least one edge entering the vertex and at least one edge leaving the vertex; all the other ones are eliminated with the adjoining edges.

By definition, each of the edges $\hat{b}_{ij}^k$ of G_σ corresponds to the map $T_j^{-1} \circ T_0^{-k} : \sigma_i \to \sigma_j$ which is saddle on σ_i by virtue of condition (6). Due to [Afraimovich & Shil'nikov, 1973; Afraimovich, *et al.*, 1983] we arrive at the following theorem.

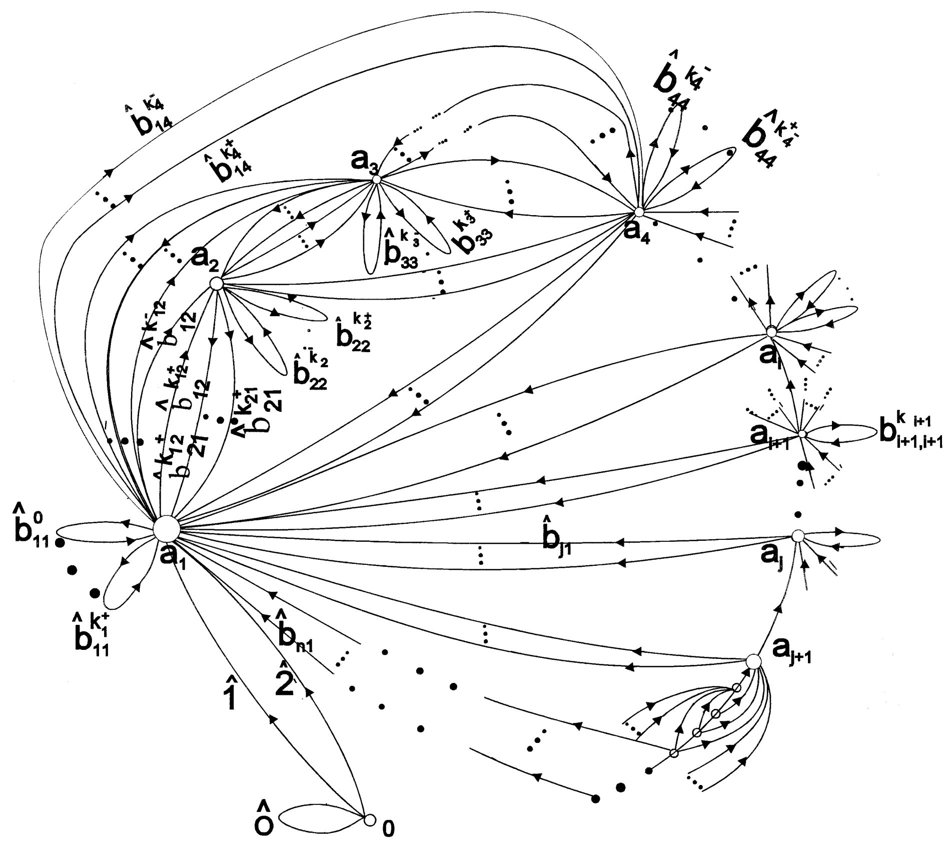


Fig. 11. The graph G_σ.

Theorem 3.1. *The system X_μ has a nontrivial hyperbolic set which is in one-to-one correspondence with the set of infinite paths along the edges of G_σ.*

The next theorem shows the nontrivial character of the bifurcational set on the parameter μ plane.

Theorem 3.2. *There exists a countable set of bifurcational curves corresponding to the presence of homoclinic loops of the saddle-focus O and a Cantor set of bifurcational curves corresponding to the situation where the one-dimensional separatrix of the saddle-focus lies on the stable manifold of a nontrivial hyperbolic set.*[3]

Proof. Let μ_0 be the value of μ at which $W^u(M_{2k})$ has a tangency with $W^s(0)$. Let μ vary so that the point $p_1(\mu)$ traces a vertical line v: $\mu^{(2)} = 0$, $\mu_0 < \mu < \mu^*$ such that on the upper bound of the line the point $p_1(\mu)$ belongs to the preimage of the upper bound of S_i and on the lower bound of the line the point P_1 belongs to the preimage $T_0^{-\bar{k}}\gamma_i$ of the lower bound of S_i (Fig 12.) For all $\mu \in v$ we have $p_1(\mu) \in T_0^{-\bar{k}}S_i$, $i > 1$. By virtue of condition (5), $T_0^{-k}S_j \bigcap T_iS_i \neq \emptyset$, $j \geq i$. Denote $T_i^{-1} \circ T_0^{-k}$ as T_{ik}. If $p_1(\mu) \in T_0^{-k}S_i$, this map is defined on the set $\bigcup_{j=i}^{\infty} S_j$, and the fixed point $M_{i\bar{k}} = T_i^{-1} \circ T_0^{-\bar{k}} M_{i\bar{k}}$ exists in S_i. We denote the stable manifold of this point as $W^s(M_i\bar{k})$. This stable invariant fiber is a topological limit for the preimages $T_{ik}^n S_j$, $J \geq i$. Evidently, the size of the set $T_{ik}^n S_j$, $j \geq i$ tends to zero as $n \to \infty$. The preimages $T_{ik}S_j$, $j > i$ lie either below W_i^s if the map $T_0^k \circ T_i$ acts in an orientable way (i is even), or on both sides if the map $T_0^k \circ T_i$ is nonorientable (i is odd).

Suppose, for more definiteness, that $p_1(\mu)$ lies above W_i^s. Evidently, for some μ^*, $P_1(\mu^*) \in W^s(M_i\bar{k})$. Then, $T_jS_j \bigcap W^s(M_{i\bar{k}}) \neq \emptyset$ at $\mu = \mu^*$ for any $j \geq i$. Since $\mathrm{lt}_{n\to\infty} T_{ik}^n S_j = W_i^s$, there exists an integer N such that the first return map on the set $T_0^{\bar{k}} \circ T_{i\bar{k}}^N \bigcup_{j=i}^{\infty} S_j$ acts as a Smale horseshoe map on each pair of adjoining regions S_j and S_{j+1} (Fig. 11).

For μ close to μ^* a large finite number m of the horseshoes is preserved. Therefore, for all close μ, in the set $T_0^{\bar{k}} \circ T_{ik}^n(S_j \bigcup S_{j+1})$, $i \leq j \leq i+m$ there exists a Cantor set $\mathcal{M}_{\mathrm{im}}^n(\mu)$ of stable invariant fibers of a nontrivial hyperbolic set which, in turn, serve

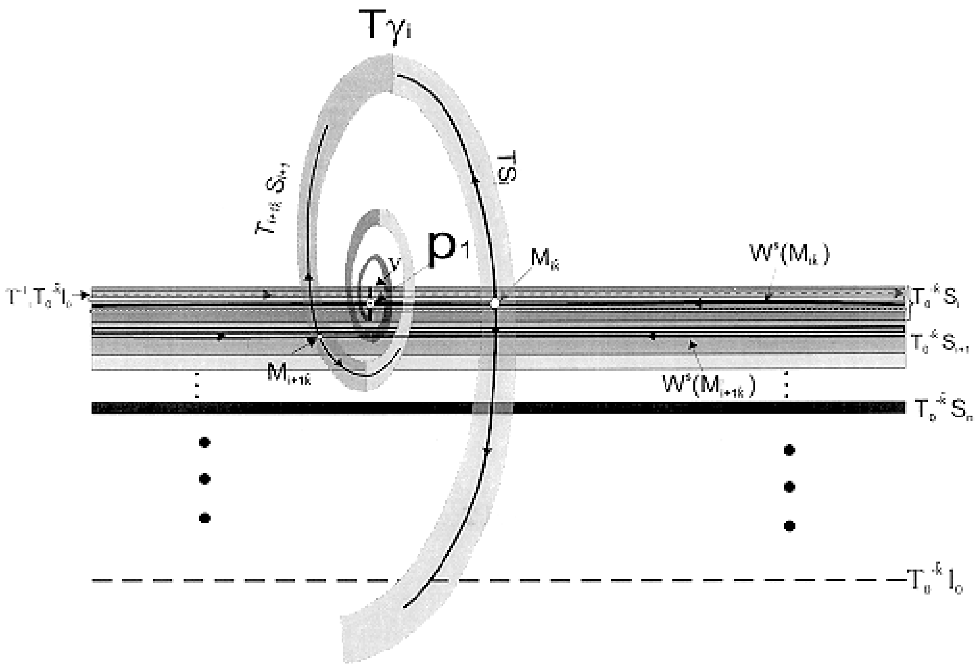


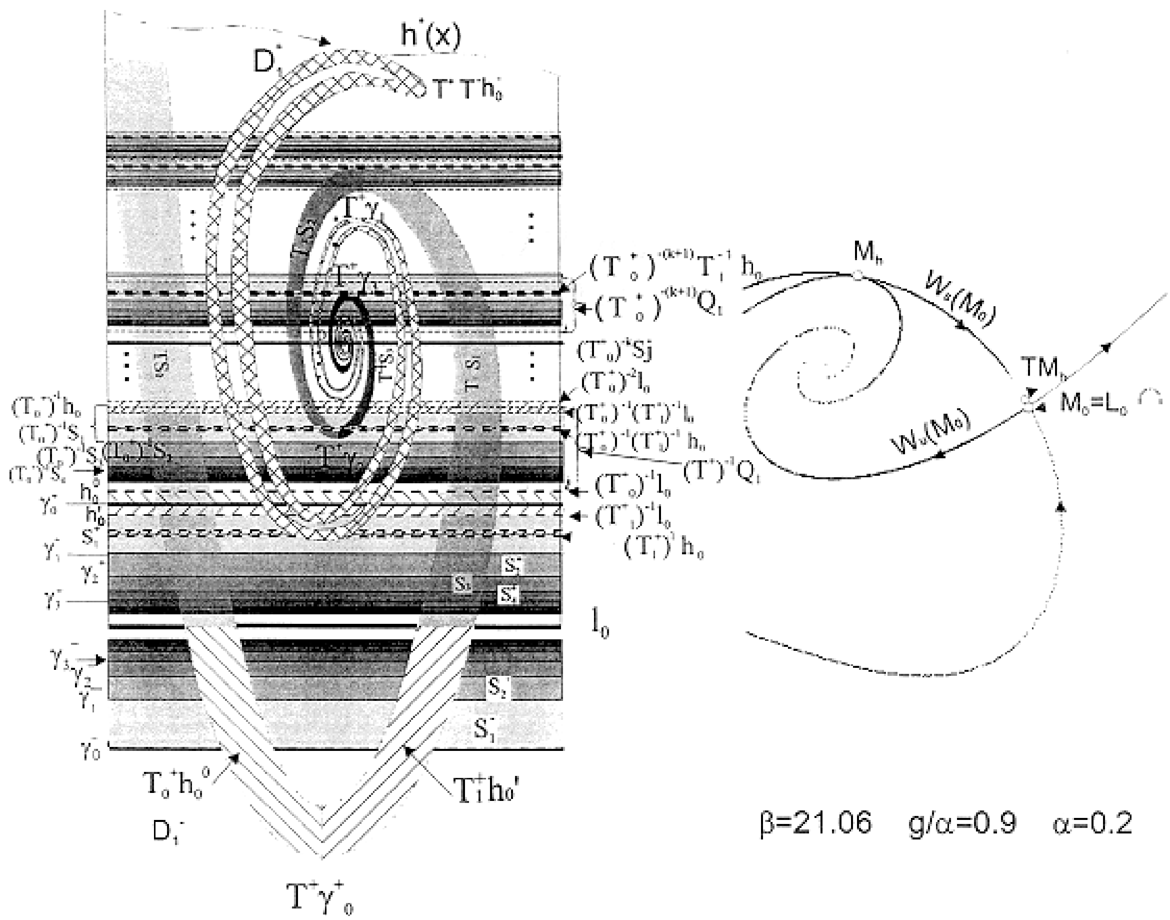
Fig. 12. The existence of infinitely many Smale's horseshoes at the moment when the one-dimensional separatrix of the saddle-focus belongs to the stable manifold of the saddle periodic orbit lying in the region $T_0^{-\bar{k}}S_i$.

[3]On the structure of the bifurcational set corresponding to homoclinic loops of systems close to a system with a saddle-focus homoclinic loop, see also [Feroe, 1993]

as limits for sequences of the lines $\mathcal{L}^n_{\mathrm{im}}$ of preimages of the discontinuity line l_0. When μ varies, the point $p_1(\mu)$ intersects all these lines and each intersection corresponds to one of the bifurcations prescribed by the theorem. The theorem is proved. ■

5. The Death of The Spiral Quasiattractor

After the bifurcation of the appearance of the heteroclinic orbit L_g that belongs to the intersection of the unstable manifold of a saddle periodic orbit and the stable manifold of the saddle-focus O, the set $T^+h_0 \in D^- \cap D_1$ (analogous to the set h_0^- in D^-) will be mapped into $D^+ \cap D_2$ and its image by the map T^+ will have a spiral shape, winding to the point $p_1(\mu)$, and will lie in $D^+ \cap D_1$. When the distance to the bifurcational set in the parameter space that corresponds to the birth of the heteroclinic orbit L_g grows, the curves $T_0^+\gamma_0^+$ and $T_0^-\gamma_0^-$ will move far from the line l_0. Consequently, the set $h_0^+(h_0^-)$ will contain more and more preimages of the regions $S_j^- \bigcap D_1(S_j^+ \bigcap D_2)$ corresponding to decreasing j. Finally, a structurally unstable heteroclinic orbit appears at the points of which the unstable manifold of the point M_{02} has a tangency with the stable manifold of the symmetric periodic orbit L (Fig. 13). After that, the region D is no longer an absorbing domain because there appear regions in D for the points of which the stable periodic orbit Γ_0 is the limit set. These regions are the preimages of the region $G_0 \in h_0$ bounded by the curves $T_0^{-1}(S_0 \bigcap W^s(L_0)$ и $T_1^{-1}(S_1 \bigcap W^s(L_0))$.



(a) (b)

Fig. 13. The moment of death of the double-scroll. Here, a heteroclinic orbit appears which tends to L_0 as $t \to \infty$ and tends, as $t \to -\infty$, to a saddle periodic orbit whose unstable manifold is the boundary of the nontrivial hyperbolic set. M_0 is the fixed point; $W^s(M_0)$ is the stable manifold of the fixed point M_0 $W^u(M_0)$ is the unstable manifold of the fixed point M_0; The points lying upper $W^s(M_0)$ tend to stable cycle Γ_0; M_h is the closed point to the nonrough heteroclinic point; TM_h is the following iteration of the point M_h.

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