A DISCRETE–TIME APPROACH TO THE STEADY STATE ANALYSIS OF DISTRIBUTED NONLINEAR AUTONOMOUS CIRCUITS

J. Bonet, P. Palà and J.M. Miró
bonet@tsc.upc.es

UPC–Department of Signal Theory and Communications
Avda. Bases de Manresa 61–73, Manresa 08240, Catalonia, Spain

ABSTRACT

We present a new method for the steady state analysis of autonomous circuits with transmission lines and generic nonlinear elements. With the temporal discretization of the equations that describe the circuit, we obtain a nonlinear algebraic formulation where the unknowns to be determined are the samples of the variables directly in the steady state, along with the oscillation period, the main unknown in autonomous circuits. An efficient scheme to build the Jacobian matrix with exact partial derivatives with respect to the oscillation period and with respect to the samples of the unknowns is described. To illustrate the proposed technique, the time–delayed Chua's circuit is analized in its periodic zones.

1. INTRODUCTION

Several tecniques have been developed to determine the steady state response of nonlinear autonomous circuits.

Shooting methods, developed in the time domain, try to determine the circuit initial conditions that make zero the transient response. In circuits which incorporate time delay these conditions must be determined with a length equal to the maximum time delay in the circuit. In autonomous circuits, the *a priori* ignorance of the oscillation period is an added difficulty.

Methods developed in the frequency domain take advantage of the circuit division into a linear and a nonlinear part to efficiently solve the linear part. However, the nonlinear part is generally best evaluated in the time domain, which makes it necessary to take successive transformations between the two domains.

In this paper we extend the discrete–time approach proposed in [1] to nonlinear autonomous circuits with transmission lines. The method is based on the formulation of the steady state equations that describe the circuit in the time domain, without any additional transformation. After discretizing these equations approximating the derivatives and the time delays by means of a linear combination of the samples of the discretized variables, we get an equivalent formulation of the problem in matrix form. The partial derivatives of the resulting equations with respect to the samples of the circuit variables to be determined and the oscillation period, the main unknown in autonomous circuits, are obtained in an exact analytic form, allowing the efficient implementation of globally convergent resolution tecniques based on modifications of Newton's method.

2. EQUATIONS FORMULATION

Consider an autonomous circuit where all the bias sources, nonlinear elements and transmission lines have been extracted. The simplified case, depicted in Fig. 1, with only one bias source, one nonlinear element and one transmission line is studied to achieve a greater insight in the formulation of the equations. The generalization for a circuit with an arbitrary number of these elements does not involve, conceptually, any added difficulty.

Since the biport resulting from the extraction of the bias source and the nonlinear element is linear, we may apply superposition in the transformed domain, expressing the control variable x of the nonlinearity in the form

$$X(s) = H_1(s,e^{s\tau})F(X) + H_2(s,e^{s\tau})V_b(s) \qquad (1)$$

and the desired output variable as

$$Y(s) = H_3(s,e^{s\tau})F(X) + H_4(s,e^{s\tau})V_b(s) \qquad (2)$$

where $H_k(s,e^{s\tau}) = -N_k(s,e^{s\tau}) / D(s,e^{s\tau})$. With this notation we rewrite (1) as

$$D(s,e^{s\tau})X(s) + N_1(s,e^{s\tau})F(X) + N_2(s,e^{s\tau})V_b(s) = 0 \qquad (3)$$

It is worth emphasizing that $N_k(s)$ and $D(s)$ are bivariate polynomials [2] of the following kind

$$P(s,e^{s\tau}) = \sum_{i=0}^{n}\sum_{k=0}^{2} p_{ik} s^i e^{-sk\tau} \qquad (4)$$

being n the order of the lumped linear multiport and τ the delay of the transmission line.

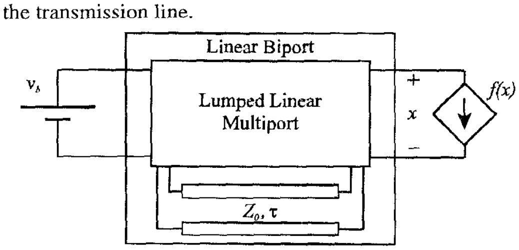

Figure 1. Simplified distributed nonlinear autonomous circuit with only one bias source, one nonlinear element and one transmission line.

The polynomial $P(s,e^{s\tau})$ applied to the Laplace transform $U(s)$ of a generic variable $u(t)$ can be expressed in the time domain as

$$P(s,e^{s\tau})U(s)\xrightarrow{ILT}\sum_{i=0}^{n}\sum_{k=0}^{2}p_{ik}\frac{d^i(u(t-k\tau))}{dt^i}. \qquad (5)$$

The discretization of (5) approximating the derivatives and the time delays by means of a linear combination of the samples of $u(t)$, and the imposition of a periodic solution of period T, will alow us to express (3) in matrix form and thus obtain a nonlinear algebraic system of N equations and N+1 unknowns: the period T and the N samples of the control variable x equally spaced $\Delta=T/N$.

3. EQUATIONS DISCRETIZATION

First we will discretize the operator defined by (5), and then we will extend the result to the discretization of (3), made up of terms formally equivalent to (5).

In the g–order Gear method the derivative is approximated at the instant $n\Delta$ interpolating $u(t)$ by a polynomial of degree g fitted to the latest g+1 samples. Other discretizations are possible [3].

Since the $k\tau$ seconds delayed function evaluated at the instant $n\Delta$, i.e. $u(n\Delta-k\tau)$, does not generally coincide with one of the samples, its value is obtained interpolating $u(t)$ by a polynomial of degree g fitted to the sample subsequent to the instant $n\Delta-k\tau$ and the g previous samples.

Thus, defining the vector of the samples of $u(t)$

$$\mathbf{u}=\left[u_1,u_2,\cdots,u_N\right]^{\mathrm{T}} \text{ whith } u_n=u(n\Delta)$$

we can compute the vector of the samples of $\frac{d(u(t))}{dt}$

$$\dot{\mathbf{u}}=\left[\dot{u}_1,\dot{u}_2,\cdots,\dot{u}_N\right]^{\mathrm{T}} \text{ from}$$

$$\dot{u}_n=\left.\frac{d(u(t))}{dt}\right|_{t=n\Delta}=\sum_{r=0}^{g}c'_r\,u_{n-r} \qquad (6)$$

and the vector of the samples of $u(t-k\tau)$

$$\mathbf{u}_k=\left[u_{k1},u_{k2},\cdots,u_{kN}\right]^{\mathrm{T}} \text{ from}$$

$$u_{kn}=u(n\Delta-k\tau)=\sum_{r=0}^{g}d'_{kr}\,u_{n-q_k-r} \qquad (7)$$

where q_k is defined according to Δ and τ as

$$q_k\Delta\le\tau<(q_k+1)\Delta$$

and c'_r and d'_{kr} are obtained from the polynomial fitting procedure described above. For the subsequent calculation of the Jacobian matrix, the dependence of the coefficients c'_r and d'_{kr} on the period T must be stated explicitly. This dependence turns out to be

$$c'_r=\frac{1}{\Delta}c_r=\frac{N}{T}c_r \text{ and } d'_{kr}=\sum_{j=0}^{g}d_{rj}(e_k)^j$$

where $e_k=\dfrac{\tau-q_k\Delta}{\Delta}=N\dfrac{\tau}{T}-q_k$, c_r and d_{rj} depend only on the order of the Gear discretization used.

Applying the discretizations (6) and (7), each operation defined in (5) can be written as the product of a matrix $\mathbf{P}(T)$ by a vector $\mathbf{u}$. Indeed, we can compute the derivative of $u(t)$ (i=1, k=0 in (5)) as

$$\frac{d(u(t))}{dt}\xrightarrow{g,N}\dot{\mathbf{u}}=\mathbf{P}_{10}(T)\mathbf{u}$$

where we define

$$\mathbf{P}_{10}(T)=\mathrm{circ}(c'_0,c'_1,\cdots,c'_g,0_{g+1},\cdots,0_{N-1})^{\mathrm{T}}=\frac{\mathbf{C}}{\Delta} \qquad (8)$$

and $\mathbf{C}$, independent of T, comes from

$$\mathbf{C}=\mathrm{circ}(c_0,c_1,\cdots,c_g,0_{g+1},\cdots,0_{N-1})^{\mathrm{T}} \qquad (9)$$

with the notation.

$$\mathrm{circ}(a_0,a_1,\cdots,a_{N-1})^{\mathrm{T}}=\begin{bmatrix}a_0 & a_{N-1} & \cdots & a_1\\ a_1 & a_0 & \cdots & a_2\\ \vdots & \vdots & \ddots & \vdots\\ a_{N-1} & a_{N-2} & \cdots & a_0\end{bmatrix}.$$

We can also compute $u(t)$ delayed $k\tau$ seconds as (i=0, k in (5))

$$u(t-k\tau)\xrightarrow{g,N}\mathbf{u}_k=\mathbf{P}_{0k}(T)\mathbf{u}$$

where

$$\mathbf{P}_{0k}(T)=\mathrm{circ}(0_0,\cdots,0_{q_k-1},d'_{k0},\cdots,d'_{kg},0_{q_k+g+1},\cdots,0_{N-1})^{\mathrm{T}} \qquad (10)$$

and where the coefficients d'_{kr} can be written as a polynomial in the above defined e_k

$$\begin{bmatrix}d'_{k0}\\ d'_{k1}\\ \vdots\\ d'_{kg}\end{bmatrix}=\begin{bmatrix}d_{00} & d_{01} & \cdots & d_{0g}\\ d_{10} & d_{11} & \cdots & d_{1g}\\ \vdots & \vdots & \ddots & \vdots\\ d_{g0} & d_{g1} & \cdots & d_{gg}\end{bmatrix}\cdot\begin{bmatrix}(e_k)^0\\ (e_k)^1\\ \vdots\\ (e_k)^g\end{bmatrix}$$

or in compact form

$$\mathbf{d}_k(T) = \mathbf{D}\mathbf{e}_k(T) \tag{11}$$

the matrix **D** being independent of both T and k.

Finally, the discretization of (5) results in the matrix

$$\mathbf{P}(T)\mathbf{u} = \sum_{i=0}^{n}\sum_{k=0}^{2} p_{ik}\mathbf{P}_{ik}(T)\mathbf{u} \tag{12}$$

where it is possible to decompose $\mathbf{P}_{ik}(T)$ in terms of the matrices defined in (8) and (10) as

$$\mathbf{P}_{ik}(T) = \mathbf{P}_{i0}(T)\mathbf{P}_{0k}(T) = (\mathbf{P}_{10}(T))^i\mathbf{P}_{0k}(T)\cdot \tag{13}$$

Multiplying by i matrices $\mathbf{P}_{10}(T)$ and by $\mathbf{P}_{0k}(T)$ corresponds to taking the i–th derivative and delaying the function $k\tau$ seconds. Since these matrices are circulant, their product is commutative and the result of this product is another circulant matrix. Moreover, there are only g+1 nonzero elements in each row. These properties are significant because they allow us to solve the resulting system of equations, to be described next, at a little computational cost.

4. RESULTING SYSTEM OF EQUATIONS

If we apply this idea to each of the products that appear in (3), we obtain an equivalent formulation in the form

$$\mathbf{D}(T)\mathbf{x} + \mathbf{N}_1(T)\mathbf{f}(\mathbf{x}) + \mathbf{N}_2(T)\mathbf{v}_\mathbf{b} = \mathbf{0} \tag{14}$$

where each matrix, once the order of discretization to be used has been chosen, only depends on T. The matrices $\mathbf{N}_1(T)$, $\mathbf{N}_2(T)$ and $\mathbf{D}(T)$ are a linear combination of $\mathbf{P}_{ik}(T)$, similar in form to $\mathbf{P}(T)$ defined in (12).

Since in autonomous circuits the period T is unknown, the system (14) has infinite solutions, which only differ on an arbitrary time delay. To avoid this problem, one of the samples of the control variable x is fixed to a value which, *a priori*, the solution is expected to take. Thus, from now on we will assume that the first sample of x is known.

In the case of autonomous circuits the vector $\mathbf{v}_b$ comes from the bias sources, i.e. all the samples have the same value. Thus, the derivative operator on $\mathbf{v}_b$ is zero and the time delay is a neutral operator. So, $\mathbf{N}_2(T)\mathbf{v}_b = b\mathbf{v}_b$, where b is a constant.

5. COMPUTATION OF THE JACOBIAN MATRIX

Efficiently solving (14) requires to use globally convergent methods based on Newton's method [4]. So we need to know the dependence of each one of the N equations with respect to the N unknowns of the system (T, x_2, x_3, ... x_N). To compute the derivative with respect to the period T, we will previously compute the Jacobian matrix of (12). Since the samples of $u(t)$ do not depend on the period T

$$\frac{d(\mathbf{P}(T)\mathbf{u})}{dT} = \sum_{i=0}^{n}\sum_{k=0}^{2} p_{ik}\frac{d(\mathbf{P}_{ik}(T))}{dT}\mathbf{u}\cdot \tag{15}$$

Using (13) and the chain rule we express

$$\frac{d(\mathbf{P}_{ik}(T))}{dT} = \frac{d(\mathbf{P}_{i0}(T))}{dT}\mathbf{P}_{0k}(T) + \mathbf{P}_{i0}(T)\frac{d(\mathbf{P}_{0k}(T))}{dT}$$

or in compact form

$$\dot{\mathbf{P}}_{ik}(T) = \dot{\mathbf{P}}_{i0}(T)\mathbf{P}_{0k}(T) + \mathbf{P}_{i0}(T)\dot{\mathbf{P}}_{0k}(T)\cdot \tag{16}$$

The computation of the derivative that appears in the first product is immediate since

$$\dot{\mathbf{P}}_{i0} = \frac{d(\frac{1}{\Delta^i}\mathbf{C}^i)}{dT} = \frac{-i}{T}\mathbf{P}_{i0}\cdot \tag{17}$$

For the computation of the derivative which appears in the second product, we must recall the dependence of $\mathbf{P}_{0k}(T)$ on $\mathbf{e}_k(T)$ according to (10) and (11). First we define

$$\dot{\mathbf{P}}_{0k}(T) = \mathrm{circ}(0_0,\cdots,0_{q_k},\dot{d}'_{k0},\cdots,\dot{d}'_{kg},0_{q_k+g+2},\cdots,0_{N-1})^\mathrm{T} \tag{18}$$

where

$$\dot{\mathbf{d}}_k(T) = \left[\dot{d}'_{k0},\dot{d}'_{k1},\cdots,\dot{d}'_{kg}\right] = \frac{d(\mathbf{d}_k(T))}{dT}\cdot \tag{19}$$

Now, using (11)

$$\dot{\mathbf{d}}_k(T) = \mathbf{D}\frac{d(\mathbf{e}_k(T))}{dT} = \frac{-1}{T}\mathbf{D}\mathbf{Q}_k\mathbf{e}_k(T) \tag{20}$$

where

$$\mathbf{Q}_k = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ q_k & 1 & 0 & \ddots & \vdots \\ 0 & 2q_k & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & gq_k & g \end{bmatrix} \tag{21}$$

Once the matrices $\dot{\mathbf{P}}_{ik}(T)$ have been computed, the computation of $\dot{\mathbf{N}}_1(T)$, $\dot{\mathbf{N}}_2(T)$ and $\dot{\mathbf{D}}(T)$ is straightforward since $\mathbf{N}_1(T)$, $\mathbf{N}_2(T)$ and $\mathbf{D}(T)$ of (14) are linear combination of the matrices $\mathbf{P}_{ik}(T)$. Thus, the first column of the Jacobian matrix is expressed analitically

$$\mathbf{J}(:,1) = \dot{\mathbf{D}}(T)\mathbf{x} + \dot{\mathbf{N}}_1(T)\mathbf{f}(\mathbf{x}) + \dot{\mathbf{N}}_2(T)\mathbf{v}_b\,. \tag{22}$$

The rest of the columns of the Jacobian matrix are easily determined since only the vectors **x** and **f(x)** depend on the N–1 unknown samples, and their partial derivatives are immediate. So, the remaining columns of the Jacobian matrix are expressed analytically

$$\mathbf{J}(:,2{:}N) = \mathbf{D}(:,2{:}N) + \mathbf{N}_1(:,2{:}N)\mathbf{F}'(\mathbf{x}) \tag{23}$$

with

$$\mathbf{F}'(\mathbf{x}) = diag(f'(x_2), \cdots, f'(x_N)) \tag{24}$$

where

$$f'(x_k) = \left. \frac{d(f(x))}{dx} \right|_{x = x_k} \tag{25}$$

6. APPLICATION TO THE TIME–DELAYED CHUA'S CIRCUIT

The described technique has been applied to the determination of the steady state of the control variable v in the time–delayed Chua's circuit shown in Fig. 2. The normalized values of the parameters which appear in the circuit, with four significant digits, are Z_0=0.4243, τ=4.423, R=1.3 and C=1. The i–v characteristic of the nonlinearity is depicted in Fig. 3 with V_1=1, V_2=8, m_0= −0.7576, m_1= −0.4091 and m_2=4.546.

The second–order Gear discretization has been used. For this discretization the values of the coefficients c_r and d_{rq} are

$$\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -2 \\ 0.5 \end{bmatrix}; \; \mathbf{D} = \begin{bmatrix} 1 & -1.5 & 0.5 \\ 0 & +2 & -1 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

The initialization of the iterative process has been made with N=64 samples of a sinusoidal signal of period T_i=15 s and amplitude A_i=10 V, obtaining the waveform depicted in Fig. 4 of period T=20.09 s and maximum symmetric amplitude A=8.638 V which corresponds to a limit circle in the phase plane. The results coincide with those obtained using PSpice.

7. CONCLUSIONS

A new method to determine the steady state response of nonlinear autonomous circuits with distributed parameters has been presented. The method is based on the time–domain discretization of the equations that describe the circuit, transforming the initial problem, the solution of a nonlinear difference differential system of equations, into the solution of a nonlinear algebraic system of equations.

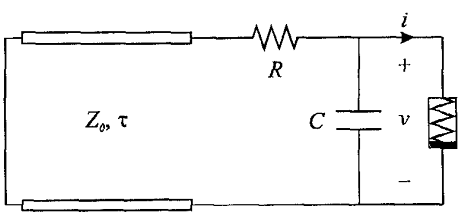

Figure 2. A modification of the Chua's circuit that results in the time–delayed Chua's circuit.

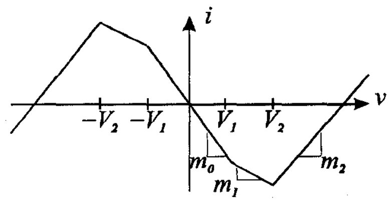

Figure 3. The i–v characteristic of the piecewise linear resistor of the time–delayed Chua's circuit.

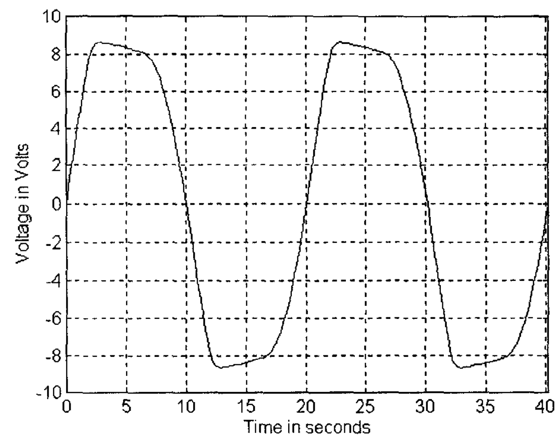

Figure 4. Two periods of the waveform obtained using the described algorithm, which corresponds to a limit circle in the phase plane.

To efficiently solve the obtained system of equations, globally convergent algorithms based on Newton's method have been implemented. The exact analytic computation of the required partial derivatives has been described in detail.

To validate the method, it has been applied to the determination of the steady state response of the time–delayed Chua's circuit in one of its periodic windows, a paradigmatic example of the kind of circuits to which this paper refers. The results coincide with those obtained using integration techniques, without having to compute the response until the transient dies out.

In the future, we intend to extend the described method to allow the inclusion of RLCG transmission lines [5] with frequency dependent parameters.

8. REFERENCES

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