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Birth of double–double scroll attractor in coupled Chua circuits

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Abstract

We discuss the creation of hyperchaotic attractors in a system of two coupled Chua circuits. Both mutual and unidirectional couplings are considered. Results from chaos synchronization theory allow us to determine chaos–hyperchaos intermittency.

1. Introduction

Hyperchaotic behaviour is characterized by two positive Lyapunov exponents. In dissipative systems hyperchaotic attractors can arise in at least four-dimensional systems of ordinary differential equations [1-12]. There are a number of dynamical systems evolving on hyperchaotic attractors, for example the chemical reaction model [1,2], electronic circuits [6], coupled logistic maps [7] and generalized Hénon maps [8]. Recently, hyperchaotic attractors have been experimentally observed in hydrodynamics [9], semiconductor systems [10] and a chain of Chua circuits [11]. Transition from chaos to hyperchaos in nonautonomous systems of periodically or chaotically coupled oscillators has been investigated in Refs. [3,5].

In what follows we investigate the hyperchaotic attractors in an autonomous system of a pair of coupled identical Chua circuits, whose combined equations of motion are

$$\dot{x}=\alpha(y-x-f(x)) , \tag{1a}$$

$$\dot{y}=x-y+z+K(v-y) , \tag{1b}$$

$$\dot{z}=-\beta y , \tag{1c}$$

$$\dot{u}=\alpha(v-u-f(u)) , \tag{1d}$$

$$\dot{v}=u-v+w+M(y-v) , \tag{1e}$$

$$\dot{w}=-\beta v , \tag{1f}$$

where

$$f(x)=bx+\tfrac{1}{2}(a-b)(|x+1|-|x-1|) ,$$
$$f(u)=bu+\tfrac{1}{2}(a-b)(|u+1|-|u-1|) , \tag{2}$$

α, β, a and b are constants. The first Chua circuit (Eqs. (1a)–(1c)) is coupled to the second one (Eqs. (1d)–(1f)) in such a way that the difference between the signals y and v

$$d_1(t)=K(y-v) , \tag{3}$$

or

$$d_2(t)=M(v-y) , \tag{4}$$

is respectively introduced into the first or the second circuit as a negative feedback. K, $M>0$ are the stiffness of the perturbations. If K and M are nonzero the two circuits are mutually coupled and if one of them is zero both circuits are coupled unidirectionally.

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The Chua circuit is of course a particular dynamical system, but a great number of extensive studies of it dynamics showed that it can be considered as paradigms for chaos [13]. That is why we believe that our results presented in this Letter can be generalized for a larger class of dynamical systems.

The outline of this Letter is as follows. Section 2 describes the complete route to hyperchaos in mutually ($K=M$) coupled Chua circuits. In Section 3 we consider the birth of hyperchaotic attractors in a unidirectionally coupled system and we describe chaos–hyperchaos intermittency. This new type of intermittent behaviour is easily observed in unidirectionally coupled circuits. Finally, we summarize our results in Section 4.

2. Mutual coupling

Considering the case of mutual coupling it is necessary to note that for $K=M$ Eqs. (1) are symmetrical with respect to the variable transposition

$$(x, y, z, u, v, w) \to (-x, -y, -z, -u, -v, -w)\,,$$
$$(x, y, z, u, v, w) \to (u, v, w, x, y, z)\,,$$
$$(x, y, z, u, v, w) \to (-u, -v, -w, -x, -y, -z)\,, \tag{5}$$

and that they have the following nine fixed points,

P_1: $(x=D, y=0, z=-D, u=D, v=0, w=-D)$,

P_2: $(x=D, y=0, z=-D, u=-D, v=0, w=D)$,

P_3: $(x=-D, y=0, z=D, u=-D, v=0, w=D)$,

P_4: $(x=-D, y=0, z=D, u=D, v=0, w=-D)$,

P_5: $(x=0, y=0, z=0, u=0, v=0, w=0)$,

P_6: $(x=0, y=0, z=0, u=-D, v=0, w=D)$,

P_7: $(x=0, y=0, z=-D, u=D, v=0, w=-D)$,

P_8: $(x=-D, y=0, z=D, u=0, v=0, w=0)$,

P_9: $(x=D, y=0, z=-D, u=0, v=0, w=0)$,

where

$$D=\frac{a-b}{b+1}\,.$$

In our numerical investigation we considered the following parameter values: $\beta=22.0$, $a=-1.1428571$, $b=-0.71428571$ and $K=M=0.0025$, and α was taken as a control parameter. Numerical computations have been performed using the fourth-order Runge–Kutta method. For $\alpha<8.78$, the fixed points P_1, P_2, P_3 and P_4 are stable. When crossing the line $\alpha=8.78$ on the parameter plane, limit cycles with period T are born from these points due to Hopf bifurcation. The phase trajectory projections of these cycles on the u–x plane are schematically shown in Fig. 1. Cycles C_1^0, C_3^0 as well as C_2^0, C_4^0 are symmetrical with respect to transposition (5). Two of them (C_1^0, C_3^0) lie in the symmetric subspace ($x=u$, $y=v$, $z=w$) of the complete phase space (they are located on the plane normal to the plane of Fig. 1 and that is why they are visible as lines) and two of the others are placed outside this subspace. It is necessary to note that the cycles C_2^0 and C_4^0 are self-symmetric with respect to substitution (5) in the sense that a point on the cycle transforms to a point on the same cycle under this substitution. A detailed description of the dynamics of two mutually coupled Chua circuits with a full spectrum of Lyapunov exponents in the considered interval $\alpha<14.0$ is given elsewhere [35]. Here we describe only the properties of hyperchaotic attractors.

After a series of bifurcations [35] at $\alpha>11.8$ we observe four co-existing, symmetrical, with respect to transposition (5), hyperchaotic attractors born from the stable fixed points P_1, P_2, P_3 and P_4. An example of such an attractor is shown in Fig. 2. Phase space

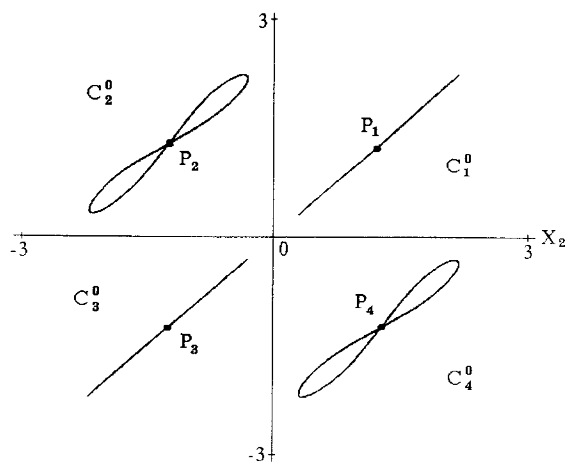

Fig. 1. Location of the fixed points in the phase space of Eqs. (1).

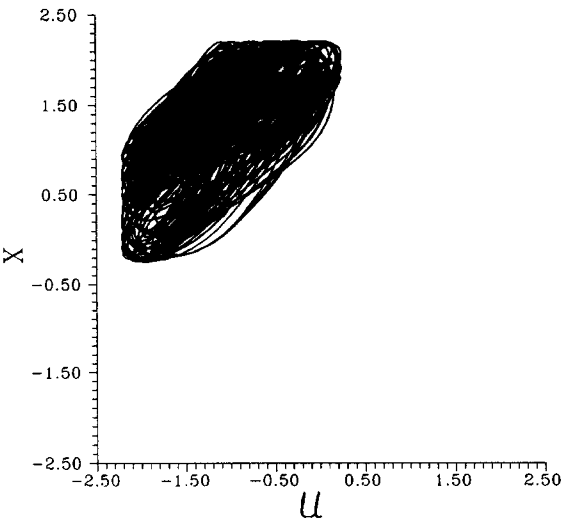

Fig. 2. Hyperchaotic attractors of Eqs. (1) (mutual coupling) shown in the 2D x–u projection; $\alpha = 11.8$; Lyapunov exponents: $\lambda_1 = 0.268$, $\lambda_2 = 0.236$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = -2.75$, $\lambda_6 = -2.82$.

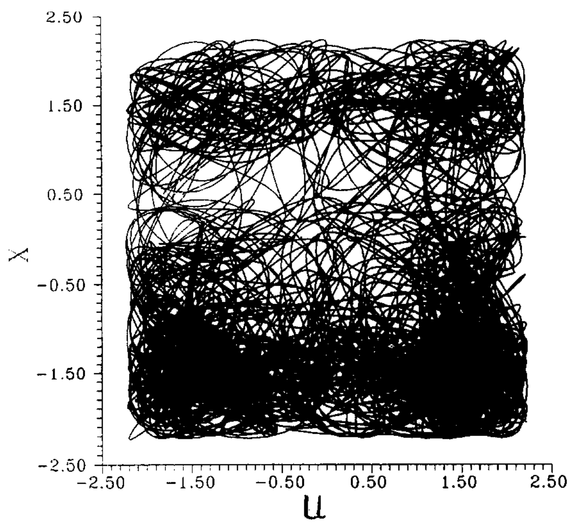

Fig. 3. Hyperchaotic attractor of Eqs. (1) (mutual coupling) shown in the 2D x–u projection; $\alpha = 12.2$; Lyapunov exponents: $\lambda_1 = 0.416$, $\lambda_2 = 0.373$, $\lambda_3 = 0$ $\lambda_4 = 0$, $\lambda_5 = -3.151$, $\lambda_6 = -3.577$.

trajectories on this attractor are characterised by two positive Lyapunov exponents ($\lambda_1 = 0.268$, $\lambda_2 = 0.236$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = -2.75$, $\lambda_6 = -2.82$) and the system evolution takes place in the neighbourhood of one of the fixed points P_1, P_2, P_3 and P_4. At a value $\alpha \approx 12.1$ all four hyperchaotic attractors of this type merge together to create a hyperchaotic attractor shown in Fig. 3. Phase space trajectories on this attractor are also characterised by two positive Lyapunov exponents ($\lambda_1 = 0.416$, $\lambda_2 = 0.373$, $\lambda_3 = 0$ $\lambda_4 = 0$, $\lambda_5 = -3.151$, $\lambda_6 = -3.577$), but unlike in the previous case they jump between neighbourhoods of P_1, P_2, P_3 and P_4 in an unpredictable way. As the mechanism of the origin of this attractor is similar to the creation of a classical double-scroll attractor in one Chua circuit [26,27] we propose to call it double-double-scroll.

3. Unidirectional coupling: chaos–hyperchaos intermittency

In the case of unidirectional coupling ($M = 0$), if we consider again α as a control parameter, we observe the same types of hyperchaotic attractors. Alternatively if we fix the value of α in such a way that both Chua circuits show chaotic behaviour and consider the coupling stiffness K as a control parameter, then results from chaos synchronization theory allow us to observe a new type of intermittency.

First let us recall some fundamental properties of the theory of intermittency. Intermittency is a type of chaotic behaviour commonly observed in deterministic systems [14–22]. It is characterized by long periods of regular motion interrupted by short chaotic bursts. When a burst starts at the end of a laminar phase this denotes an instability of the periodic motion due to the fact that the modulus of at least one Floquet multiplier is larger than one. Besides this simple case, more complicated intermittent behaviour can take place in the general case, such as the "chaos–chaos" intermittency [24,25].

The possibility of chaotic (lower-dimensional) and hyperchaotic (higher-dimensional) attractors in the six-dimensional phase space of Eq. (1) shows the possibility of chaos–hyperchaos intermittency. This new type of intermittent behaviour occurs when the system evolution takes place on a chaotic (lower-dimensional) attractor embedded in the three-dimensional subspace of the six-dimensional phase space for a significantly long time and only occasionally bursts to a higher-dimensional attractor.

In our numerical investigations we considered the following parameter values: $\alpha = 10.0$, $\beta = 14.87$, $a = -1.27$, $b = -0.68$ and $M = 0$, i.e. unidirectional coupling of both Chua circuits. In the case of $K = 0$

(no coupling) the dynamics of both Chua circuits evolves along a double-scroll attractor [26,27]. We chose slightly different initial conditions for both circuits $x(0)=0.010$, $u(0)=0.011$ $y(0)=z(0)=v(0)=w(0)=0$. Numerical computations in this section have been performed using the software INSITE [28]. Lyapunov exponents have been calculated using the algorithm described on p. 80 of Ref. [28].

The trajectories of Eqs. (1d)–(1f) are located on a 3D manifold. If the trajectories of the whole system (1a)–(1f) approach this 3D manifold (in our case this manifold is described by $x=u$, $y=v$ and $z=w$) as well (the attractor is embedded in the three-dimensional subspace of the six-dimensional phase space of Eq. (1)), then the first circuit simply reproduces the chaotic oscillations of the second circuit. In this case, all trajectories converge to the attractor of Eqs. (1d)–(1f), $d(t)\to 0$ and both circuits synchronize [29,30]. The described 3D manifold exists for any value of the coupling stiffness K. This enables us to investigate the stability of the chaotic limit set located in this manifold as a function of K. The spectrum of the Lyapunov exponents of the coupled system (1) can be divided into two subsets $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively, along the orthogonal to the manifold. The first subset of Lyapunov exponents is associated with the second circuit (1d)–(1f) and consists of three exponents describing the evolution of perturbations tangent to the manifold. The Lyapunov exponents of the second subset describe the evolution of the perturbations transverse to the manifold. If at least one $\lambda^{(2)}$-Lyapunov exponent is positive the resulting limit set is not restricted to the manifold of the second circuit (1d)–(1f) and we observe a hyperchaos regime. As shown by de Sousa et al. [31] the $\lambda^{(2)}$-Lyapunov exponents are equivalent to the conditional or sub-Lyapunov exponents of Pecora and Carroll [32,33]. This is why the chaos–hyperchaos transition in our system is strictly connected with the synchronization problem.

As is was shown in Ref. [11] for smaller values of K ($K<1.17$) the chaotic trajectories of system (5) are characterized by two positive Lyapunov exponents; one in the $\lambda^{(1)}$-subset and the other in the $\lambda^{(2)}$-subset, so that in this case the two Chua double scroll circuits cannot synchronize. In this case we have hyperchaotic evolution of the system. For higher values of K ($K>1.17$) there is no positive Lyapunov exponent in the $\lambda^{(2)}$-subset, the evolution takes place on a three-dimensional manifold and the circuits can synchronize. The synchronization property of the chaotic attractors in our case allows us to find the qualitative difference between chaotic and hyperchaotic attractors from in x–u–z projections. Generally, this distinction is not straightforward [4].

In Fig. 4 we show three-dimensional x–u–z projections of chaotic and hyperchaotic attractors. The evolution of the projection of the chaotic attractor of Fig. 4a takes place on a two-dimensional $x=u$ plane, while the evolution of the projection of the hyperchaotic attractor of Fig. 4b is represented by a three-dimensional structure. The attractor of Fig. 4a is a classical double-scroll attractor, while the attractor of Fig. 4b has the same structure as the double–double scroll attractor introduced in the previous section.

The same x–u–z projections allow us to observe chaos–hyperchaos intermittency. Shortly after the transition from chaos to hyperchaos at $K=1.17$ the trajectories of system (1) evolve on a three-dimensional manifold for a long time and only occasionally burst to higher dimensions. This process can be observed in Fig. 5. In Fig. 5a we observe how after a relatively long evolution on a three-dimensional manifold the trajectory goes out of it towards one of the unstable fixed points. Fig. 5b shows the double–double scroll attractor shortly after its birth. With a further decrease of the coupling stiffness K the intervals with evolution on the three-dimensional manifold become shorter and finally at $K=0.92$ they disappear.

The chaos–hyperchaos intermittency phenomenology is as follows. For values of the control parameter p less than a critical transition value p^* the attractor is chaotic and the system trajectories evolve on a 3D manifold. For p slightly larger than p^* there are long stretches of time ("chaotic phases") during which the trajectory appears to evolve on the 3D manifold and closely resembles the trajectory for $p<p^*$, but this chaotic behaviour is intermittently interrupted by a finite duration burst in which the trajectory leaves the 3D manifold of the chaotic attractor. These bursts occur at seemingly random times, but one can define an average length of time interval during which the system trajectory evolves on the 3D-manifold (a mean time between the bursts to a higher

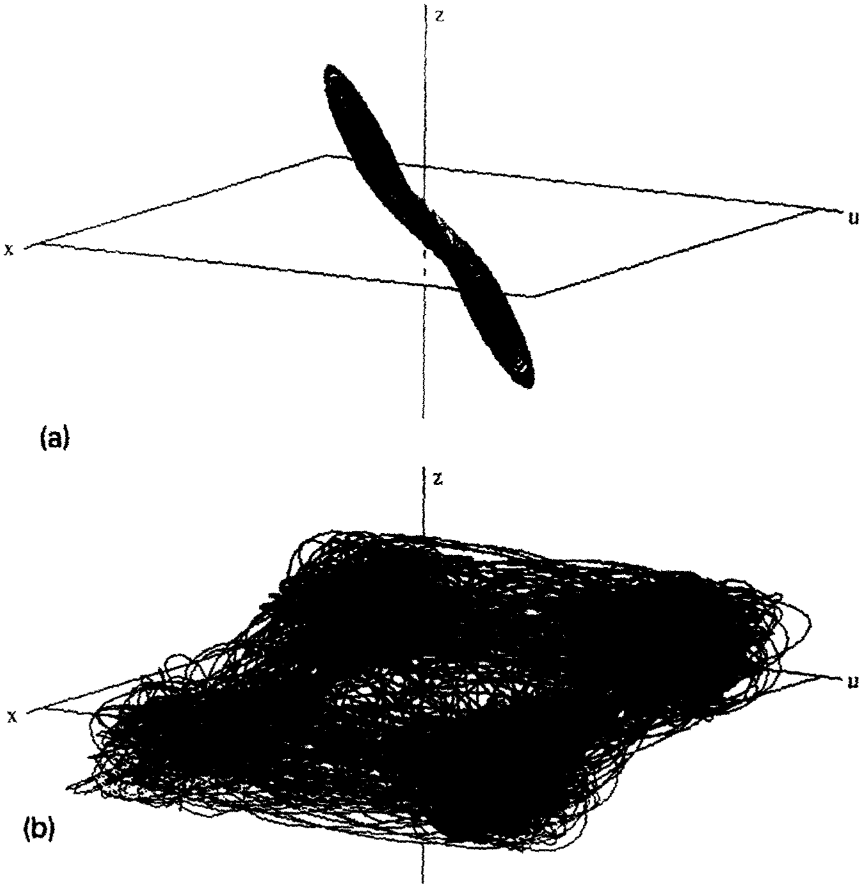

Fig. 4. Chaotic and hyperchaotic attractors of Eqs. (1) (unidirectional coupling) shown in the 3D x–u–z projection. (a) $K=1.15$: chaotic attractor; (b) $K=0.02$: hyperchaotic attractor; $\lambda_1=0.431$, $\lambda_2=0.412$, $\lambda_3=0$ $\lambda_4=0$, $\lambda_5=-3.741$, $\lambda_6=-3.852$.

dimensional manifold) $T(p)$. As p increases substantially above p^*, the bursts become so frequent that the chaotic evolution on the 3D-manifold can no longer be distinguished.

In our system (1) we observed the following scaling behaviour of the average interburst time $T(p)$,

$$T(p) \propto (p-p^*)^{-\alpha}, \tag{7}$$

where $p=-K$ and $\alpha=0.22\pm0.01$. Computation of Lyapunov exponents of trajectories showing intermittency with two different types of behaviour separated by long time intervals requires the consideration of long trajectories to obtain convergence to true values. Long intervals with different types of behaviour can cause fluctuations of transient Lyapunov exponents similar to those described in Ref. [37]. In this type of computations we considered trajectories which consist of 10^6 intervals with evolution on a chaotic attractor and 10^6 bursts to a higher dimensional attractor. In calculations of the scaling factor α we considered an average of 1000 simulations for randomly chosen initial conditions: $x(0)\neq u(0)\in[-0.1,0.1]$, $y(0)=z(0)=v(0)=w(0)=0$. We considered 1000 trajectories for different initial conditions to be sure that the described behaviour is typical in the domain of the phase space, as our system is quasi-hyperbolic and many different attractors can coexist.

In the regions of chaos–hyperchaos intermittency in K-parameter space one can observe exponential growth of the Lyapunov dimension of an attractor,

$$d_L=j+\frac{\sum_{i=1}^{j}\lambda_i}{|\lambda_{j+1}|}, \tag{8}$$

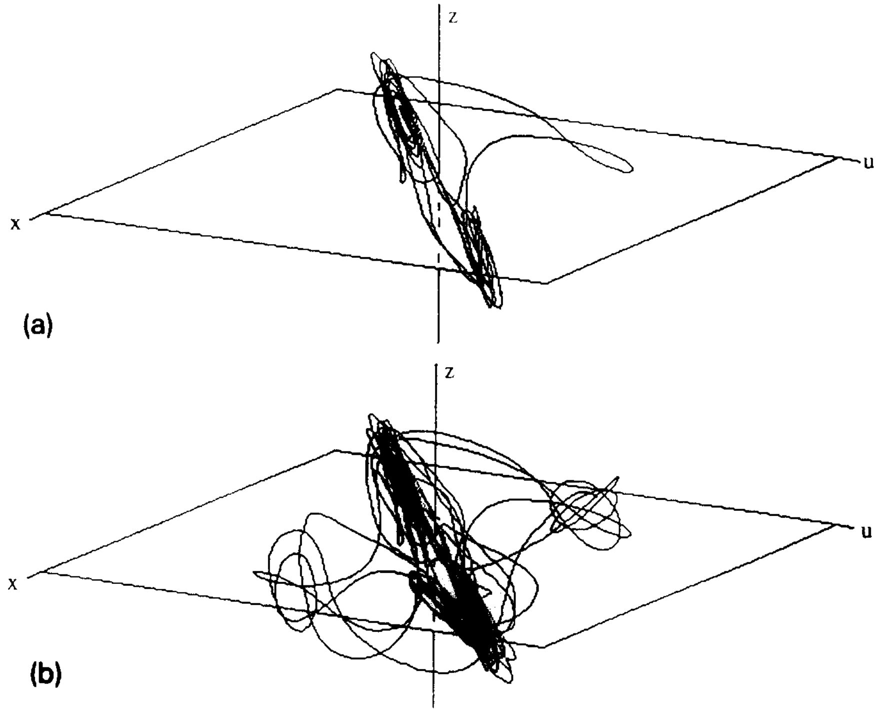

Fig. 5. Details of the evolution on the hyperchaotic attractor (unidirectional coupling); (a) escape from the 3D manifold; (b) birth of the double–double scroll attractor.

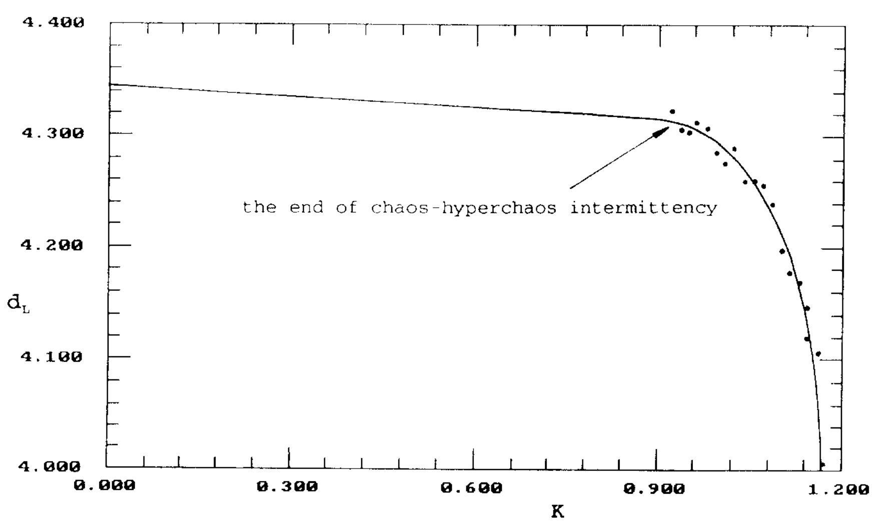

Fig. 6. Plot of the Lyapunov dimension versus K for Eqs. (1): $\alpha=10.0$, $\beta=14.87$, $a=-1.27$, $b=-0.68$; dotted line: numerical results; solid line: scaling law (6).

where j is determined by $\sum_{i=1}^{j}\lambda\geqslant 0$ but $\sum_{i=1}^{N+1}\lambda<0$ [26]. In Fig. 3 we show the plot of d_L versus K. It can be seen that with a decrease of K, the Lyapunov dimension d_L grows. We can distinguish two regions. In the first one, in which chaos–hyperchaos intermittency takes place, the growth is exponential with the scaling law

$$d_L \propto (p-p^*)^{-\alpha}, \tag{9}$$

with approximately the same α as in scaling law (4). It should be noted here that relation (6) holds as long as chaotic evolution on the 3D-manifold can be distinguished (in our system for $p \in (-1.12, -0.92)$. When hyperchaotic behaviour is fully developed (the evolution on the 3D-manifold cannot be distinguished) the Lyapunov dimension becomes much smaller and tends towards the value $2d_L^{ds}$ for $K=0$, where $d_L^{ds}=2.11$ is the information dimension of the double scroll attractor. The Lyapunov exponents (Lyapunov dimension) necessary to establish relation (9) were (was) based on the average of 1000 numerical simulations of hyperchaotic trajectories for randomly chosen initial conditions: $x(0) \neq u(0) \in [-0.1, 0.11]$, $y(0)=z(0)=v(0)=w(0)=0$. A mean-square fit was applied.

Finally it should be noted that a similar chaos–hyperchaos intermittency can be observed in the mutually coupled system considered in Section 2, but in this case it cannot be directly observed and can be determined only by investigations of power spectra [36].

4. Conclusions

To summarize, it has been demonstrated here that two coupled Chua circuits can exhibit chaotic or hyperchaotic behaviour. Hyperchaotic attractors are robust in the phase space of both mutually and unidirectionally coupled circuits. We determined two types of attractors with two positive Lyapunov exponents. One type is equivalent with the spiral Rössler-type of chaotic attractor in one Chua circuit. Depending on the initial conditions two such attractors exist in the phase space. The second type, which we called double–double-scroll attractor, is created when four spiral-type attractors merge together in a similar way as the classical double-scroll attractor is created from two spiral-type attractors in one Chua circuit.

Additionally, it has been shown that two coupled Chua double scroll circuits can demonstrate chaos–hyperchaos intermittency. This new type of intermittent behaviour is characterized by the long evolutions of the hyperchaotic trajectory on a lower-dimensional chaotic attractor with occasional bursts to higher dimensions. As this mechanism is similar to the mechanisms of classical intermittent behaviour [1,2], chaos–hyperchaos intermittency can be considered as its generalization for higher-dimensional systems.

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