Convex Relative Entropy Decay in Markov Chains

Varun Jog
EECS, UC Berkeley
Berkeley, CA-94720
Email: varunjog@eecs.berkeley.edu

Venkat Anantharam
EECS, UC Berkeley
Berkeley, CA-94720
Email: ananth@eecs.berkeley.edu

Abstract—We look at irreducible continuous time Markov chains with a finite or countably infinite number of states, and a unique stationary distribution \( \pi \). If the Markov chain has distribution \( \mu_t \) at time \( t \), its relative entropy to stationarity is denoted by \( h(\mu_t|\pi) \). This is a monotonically decreasing function of time, and decays to 0 at an exponential rate in most natural examples of Markov chains arising in applications. In this paper, we focus on the second derivative properties of \( h(\mu_t|\pi) \). In particular we examine when relative entropy to stationarity exhibits convex decay, independent of the starting distribution. It has been shown that convexity of \( h(\mu_t|\pi) \) in a Markov chain can lead to sharper bounds on the rate of relative entropy decay, and thus on the mixing time of the Markov chain. We study certain finite state Markov chains as well as countable state Markov chains arising from stable Jackson queueing networks.

Keywords: Markov chains, relative entropy, Jackson network, convex decay

I. INTRODUCTION

The theory of Markov chains is one of the most important tools in applied probability. Markov chains are used to model a variety of phenomena, encompassing physical systems, networking and queueing systems, as well as biological systems. These models are often capable of capturing the rich system dynamics, while still being amenable to analysis. In addition to modeling systems, another notable application of Markov chains is that of sampling from an intractable distribution. These sampling methods, called Markov Chain Monte Carlo (MCMC) methods, are used to solve problems roughly of the form: given a complicated distribution \( \pi \), the general strategy of MCMC methods is to construct a Markov chain on \( S \), whose stationary distribution is \( \pi \). Simulating this Markov chain for a long enough time gives a technique for sampling from the distribution \( \pi \) (approximately), and the mean of \( f \) can be approximated by many such samples. This naturally leads to the question: how long should a Markov chain be simulated before it is “close” to stationarity?

The rate of convergence to stationarity of an irreducible positive recurrent countable state continuous time Markov chain is measured in terms of the mixing time of the chain. We refer the reader to Montenegro & Tetali [1] and Levin, Peres & Wilmer [2] for a comprehensive survey on Markov chain mixing. The mixing time is defined as the time \( \tau \) at which the distance between the distribution at time \( \tau \) and the stationary distribution is smaller than some threshold, like \( 1/2e \). The distance used is typically defined via the \( L^p \) norm for some \( p \geq 1 \), as follows

\[
\|\mu/\pi - 1\|_{p,\pi}^p = \sum_{x \in S} \left| \frac{\mu(x)}{\pi(x)} - 1 \right|^p \pi(x),
\]

where \( \pi \) denotes the stationary distribution of the Markov chain, and \( \mu \) the probability distribution whose distance to \( \pi \) is being measured.

The squared \( L^2 \) norm is also known in the literature as variance, and is denoted by \( \text{Var}_\pi(\mu/\pi) \).

One advantage of working with variance is the availability of the Poincaré inequality, a functional inequality that bounds the rate of convergence of variance, and thereby leads to bounds on the \( L^2 \) mixing time. The Poincaré inequality states that for a certain optimal constant \( \lambda > 0 \), called the Poincaré constant, the following holds for all initial conditions \( \mu_0 \),

\[
\var_{\pi}(\mu_0/\pi) \leq \frac{1}{\lambda} \left. \frac{d}{dt} \var_{\pi}(\mu_t/\pi) \right|_{t=0} \tag{1}
\]

where \( \mu_t \) is the distribution of the Markov chain at time \( t \). It is not hard to see that the Poincaré inequality implies exponential convergence of variance, i.e.

\[
\var_{\pi}(\mu_t/\pi) \leq e^{-\lambda t} \var_{\pi}(\mu_0/\pi) \tag{2}
\]

In this paper, we shall look at another important measure of closeness used to study convergence of Markov chains, namely relative entropy, which is given by

\[
h(\mu|\pi) = \sum_{x \in S} \mu(x) \log \frac{\mu(x)}{\pi(x)},
\]

where the logarithm is to the natural base. Though relative entropy is not a norm, it satisfies certain norm-like properties. For instance \( h(\mu|\pi) \) is convex in \( \mu \), and \( h(\mu|\pi) \geq 0 \) for all \( \mu \), with equality if and only if \( \mu = \pi \).

Convexity in \( \mu \) implies that \( h(\mu_t|\pi) \) is a monotonically decreasing function in \( t \), as shown, for instance, in Cover & Thomas [3]. As \( t \to \infty \), the Markov chain approaches stationarity and \( h(\mu_t|\pi) \to 0 \). To get bounds on the speed of convergence of relative entropy, one needs to study the properties of \( h(\mu_t|\pi) \) and its derivatives. A modified log-Sobolev inequality (MSLI) is said to hold for some \( \alpha > 0 \) if the following holds for all initial conditions \( \mu_0 \),

\[
h(\mu_0|\pi) \leq -\frac{1}{\alpha} \left. \frac{d}{dt} h(\mu_t|\pi) \right|_{t=0} \tag{3}
\]
We can check that an MLSI implies exponential convergence of relative entropy, i.e.,
\[
h(\mu_t | \pi) \leq e^{-\alpha t} h(\mu_0 | \pi)
\]  
(4)

The inequality (3) and the best possible constant \(\alpha\) are analogous to the Poincaré inequality \(\Pi\) and the Poincaré constant \(\lambda\) respectively. For an introduction to MLSI and other functional inequalities, we direct the reader to Diaconis & Saloff-Coste \[4\] and Bobkov & Tetali \[5\].

Although techniques have been developed for bounding the Poincaré constant, bounds on the MLSI constant remain elusive. MLSI is essentially a statement about the first derivative of \(h(\mu_t | \pi)\), but there is no need to stop just at the first derivative. Caputo, Dai Pra & Posta \[6\] observe that if for some \(\kappa \geq 0\) the following inequality is satisfied,
\[
- \frac{d}{dt} h(\mu_t | \pi) \leq \frac{\kappa}{\kappa} \frac{d^2}{dt^2} h(\mu_t | \pi)
\]  
(5)

then (3) also holds with \(\alpha = \kappa\). In this way, estimates on \(\kappa\) lead to estimates on \(\alpha\).

Our work follows from a remark by Caputo et al. \[6\] wherein the authors note that inequality (5) is in general strictly stronger than (3): it not only implies exponential decay of relative entropy, but also that the decay is convex in time. While convex relative entropy decay in a Markov chain can lead to better estimates on its MLSI constant, it is interesting in itself to study under what conditions a Markov chain exhibits convex relative entropy decay. It appears that the second derivative behavior of relative entropy has not been studied much in the literature. Our goal is to shed some light on this second derivative behavior, in several Markov chains of interest in communications and networking applications.

This paper is structured as follows. In section II we introduce some notation that facilitates dealing with derivative expressions. In section III we introduce a class of reversible finite state Markov chains which exhibit convex relative entropy decay. In section IV we check for convexity of relative entropy decay in general reversible finite state Markov chains, providing some interesting examples and counterexamples. In section V we consider a class of countable state Markov chains arising from queueing networks, and examine relative entropy decay in the same. The Markov chains considered in this section are in general not reversible.

II. NOTATION

Given an irreducible continuous time Markov chain on a finite or countably infinite state space \(S\) with a stationary distribution \(\pi\), denote its distribution and density at time \(t\), by \(\mu_t\) and \(\rho_t\) respectively. Here, \(\rho_t\) is simply \(\frac{\mu_t}{\pi}\). Given a function \(f\) defined on the state space \(S\), \(\pi[f]\) is shorthand for \(\sum_{x \in S} \pi(x) f(x)\). The relative entropy of \(\mu_t\) with respect to \(\pi\) is expressed as
\[
h(\mu_t | \pi) = \pi[\rho_t \log \rho_t].
\]

Let \(\mathcal{L}\) and \(\mathcal{L}^*\) stand for the infinitesimal generators of the forward time and reversed time Markov chains respectively. Using \(\frac{d}{dt} \rho_t = \mathcal{L}^* \rho_t\), we can check that
\[
\frac{d}{dt} h(\mu_t | \pi) = \pi[(\mathcal{L}^* \rho_t) \log \rho_t].
\]  
(6)

Differentiating again, and using \(\pi[(\mathcal{L}^* f) g] = \pi[f \mathcal{L} g]\),
\[
\frac{d^2}{dt^2} h(\mu_t | \pi) = \pi \left[ (\mathcal{L}^* \rho_t) \log \rho_t + \frac{(\mathcal{L}^* \rho_t)^2}{\rho_t} \right],
\]  
(7)

\[
= \pi \left[ \mathcal{L}^* \rho_t \mathcal{L} \log \rho_t + \frac{(\mathcal{L} \rho_t)^2}{\rho_t} \right].
\]  
(8)

Expression (8) makes the double derivative expression seem more daunting that it actually is. We’ll now describe an interpretation that greatly simplifies dealing with this expression in case of reversible Markov chains. This interpretation comes from Caputo et al. \[6\]. We denote the set of allowed moves in a Markov chain by \(\Gamma\); this set consists of maps from the state space \(S\) to itself. Given a map \(\gamma \in \Gamma\) and a state \(\eta \in S\), an allowed move is a jump from \(\eta\) to \(\gamma(\eta)\). The rate of jumping from \(\eta\) to \(\gamma(\eta)\) via the move \(\gamma\) is denoted by \(c(\eta, \gamma)\). It is not hard to see that every Markov chain permits such a representation. Using this notation, we can express the action of the infinitesimal generator \(\mathcal{L}\), on a function \(f : S \to \mathbb{R}\) by
\[
\mathcal{L} f(\eta) = \sum_{\gamma \in \Gamma} c(\eta, \gamma)[f(\gamma(\eta)) - f(\eta)] := \sum_{\gamma \in \Gamma} c(\eta, \gamma) \nabla_{\gamma} f(\eta).
\]

With this notation, for reversible Markov chains \((\mathcal{L} = \mathcal{L}^*)\), (8) simplifies to
\[
\frac{d^2}{dt^2} h(\mu_t | \pi) = \sum_{\gamma, \delta, \eta} \nabla_{\delta} \rho_t(\eta) \nabla_{\gamma} \log \rho_t(\eta) \left[ \nabla_{\delta} \rho_t(\eta) \nabla_{\gamma} \log \rho_t(\eta) \right],
\]  
(9)

where \(\eta \in S\) and \(\gamma, \delta \in \Gamma\), see equation (2.7) of \[6\].

In this paper, our focus is on the sign of the double derivative. Scaling \(\pi\) or \(\rho\) by a constant does not affect this sign, which is why we shall often not bother to normalize them. We abuse notation slightly and continue denoting these unnormalized quantities by \(\pi\) and \(\rho\).

III. FINITE STATE REVERSIBLE MARKOV CHAINS WITH CONVEX RELATIVE ENTROPY DECAY

In this section we present a class of finite state reversible Markov chains, which exhibit convex relative entropy decay.

Theorem 1. Let \(G\) be a finite abelian group. Let \(\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_r\} \subseteq G\) and \(c : \Gamma \to (0, \infty)\) be such that

- \(\Gamma\) is closed under inversion
- \(\Gamma\) generates \(G\)
- For any \(\gamma \in \Gamma\), \(c(\gamma) = c(\gamma^{-1})\)

Consider a Markov chain with each state corresponding to an element of \(G\). From a state \(\eta\), the allowed moves are \(\eta \to \gamma(\eta)\), with a jump rate of \(c(\eta, \gamma) = c(\gamma \eta)\). For such a Markov chain, relative entropy decay is convex.

Proof: It is not hard to see that this Markov chain is reversible with a uniform stationary distribution. The RHS of
A. Two-state Markov chains

Consider a general 2-state Markov chain, with states $a$ and $b$. The set of allowed moves is $\Gamma = \{\gamma\}$ where $\gamma(a) = b$ and $\gamma(b) = a$. Since a 2-state chain is reversible, using equation (9) we get that the double derivative of relative entropy at a density $\rho_1$ is

$$
\frac{d^2}{dt^2} h(\mu_t | \pi) = \sum_{\eta \in \{a, b\}} \pi(\eta) c(\eta, \gamma)^2 \left[ \nabla_\gamma \rho_1(\eta) \nabla_\gamma \log \rho_1(\eta) + \nabla_\delta \rho_1(\eta)^2 \right].
$$

As $\nabla_\gamma f(\eta) \nabla_\gamma \log f(\eta) \geq 0$, the double derivative is seen to be non-negative and relative entropy decay is convex.

B. Three-state Markov chains

Equation (9) works out to

$$
\sum_{\eta, \gamma, \delta} |G|^{-1} c(\delta) c(\gamma) \left[ \rho_1(\delta \eta) - \rho_1(\eta) \right] \log \frac{\rho_1(\gamma \eta)}{\rho_1(\eta)} \\
+ \rho_1(\delta \eta) \rho_1(\gamma \eta) - \rho_1(\delta \eta) - \rho_1(\gamma \eta) + \rho_1(\eta) \right].
$$

It can be checked that this quantity is the same as the quantity one gets with each of the following substitutions in the terms inside the square brackets:

- $(\eta, \delta \eta, \gamma, \delta \gamma) \leftarrow (\delta \eta, \eta, \delta \gamma, \gamma)$
- $(\eta, \delta \eta, \gamma, \delta \gamma) \leftarrow (\delta \gamma, \gamma, \delta \eta, \eta)$
- $(\eta, \delta \eta, \gamma, \delta \gamma) \leftarrow (\gamma, \delta \gamma, \eta, \delta \eta)$

Adding the resulting four expressions (inside the square brackets), and rearranging the sum, gives sum of another four terms (related to each other by the same substitutions as above), of which the basic one is

$$
\rho_1(\eta) \log \rho_1(\eta) - \rho_1(\eta) \log \frac{\rho_1(\delta \eta) \rho_1(\gamma \eta)}{\rho_1(\eta)} + \frac{\rho_1(\delta \eta) \rho_1(\gamma \eta)}{\rho_1(\eta)} - \rho_1(\eta).
$$

Since this is of the form $a \log a - a \log b + b - a$ for some $\alpha, \beta > 0$, it is nonnegative.

**Remark 1.** The proof above is basically an unravelling of the proof of Lemma 2.3 and Corollary 2.4 of [6], which was inspired by the Bochner identity in Riemannian geometry. The proof of Lemma 2.3 and Corollary 2.4 of [6], which was inspired by the Bochner identity in Riemannian geometry. The proof has been spelt out for clarity.

**Remark 2.** Markov chains formed via random walks on many graphs (including the $n$-clique and the $n$-ring) satisfy the properties of Claim [1] Cayley graphs [7] are closely related to this class of graphs. In fact, a Cayley graph of an abelian group generated from a set closed under inversion (sometimes known in the literature as an undirected Cayley graph), is one such subclass of graphs.

IV. General reversible finite state Markov chains

A. Two-state Markov chain

It is optimistic to expect every 3-state chain to have convex relative entropy decay. But if one were to consider just reversible 3-state chains, one might expect to see convex relative entropy decay. Unfortunately, relative entropy defies this expectation. A counterexample was provided by Caputo et al. [9] which we repeat here. If we tighten conditions further by insisting that the 3-state chain be symmetric, it turns out that relative entropy decay is convex.

**Counterexample for 3-state reversible Markov chains:**

Consider a Markov chain with states \{1, 2, 3\} and rate matrix

$$
\begin{pmatrix}
-2c_1 & c_1 & c_1 \\
c_2 & -2c_2 & c_2 \\
c_3 & c_3 & -2c_3
\end{pmatrix},
$$

for some $c_1, c_2, c_3 > 0$. The stationary distribution $\pi \propto [1/c_1, 1/c_2, 1/c_3]$. Using equation (9), we can check that that

$$
\frac{d^2}{dt^2} h(\mu_t | \pi) \propto c_1 Q_1 + c_2 Q_2 + c_3 Q_3.
$$

Here we take $\Gamma$ to consist of the two cyclic permutations (123) and (132) with the obvious associated rates. The exact expression for $Q_1$ is

$$
Q_1 = (\rho_2 + \rho_3 - 2\rho_1) \log \frac{\rho_2 \rho_3}{\rho_1^2} + \frac{(\rho_2 - \rho_1)^2}{\rho_1} + \frac{(\rho_3 - \rho_1)^2}{\rho_1} + 2 \frac{(\rho_2 - \rho_1)(\rho_3 - \rho_1)}{\rho_1},
$$

and $Q_2$ and $Q_3$ have similar expressions. The key point to note is that $Q_1$ are dependent only on the density $\rho = (\rho_1, \rho_2, \rho_3)$. So if one of them, say $Q_1$, happened to be negative for a certain choice of $\rho$, we can make $c_1$ large enough and force $Q_1$ to become negative. Choosing $\rho_1 = 1, \rho_2 = 2$ and $\rho_3 = \epsilon$, $Q_1$ evaluates to

$$
Q_1 = \epsilon (2 \log 2 + \log \epsilon + \epsilon),
$$

which is negative for a small enough $\epsilon$.

**3-state symmetric Markov chains:**

Consider a Markov chain with a symmetric rate matrix given by

$$
Q = \begin{pmatrix}
-(b + c) & c & b \\
c & -a & c \\
b & a & -(a + b)
\end{pmatrix},
$$

for some $a, b, c \geq 0$, subject to irreducibility. The stationary distribution $\pi$ is uniform. We take $\Gamma$ to consist of the two cyclic permutations (123) and (132) with the obvious associated rates. Without loss of generality, let’s assume $\rho_1 \leq \rho_2 = 1 \leq \rho_3$. This order relation forces the contributions from states 1 and 3 to be non-negative. In the contribution from node 2, the non-positive terms are given by

$$
- ac(\rho_1 - \rho_2) \log \frac{\rho_3}{\rho_2} + ac(\rho_3 - \rho_2) \log \frac{\rho_1}{\rho_2} + 2 ac(\rho_3 - \rho_2)(\rho_1 - \rho_2).
$$

Notice that these terms are not dependent on $b$; $b$ occurs solely in conjunction with the positive terms. Thus we get that the double derivative is lower bounded by $a^2 \cdot \alpha + ac \cdot \gamma + c^2 \cdot \beta$, where

$$
\alpha = 2(\rho_3 - 1) \log \rho_3 + (\rho_3 - 1)^2 + \frac{(1 - \rho_3)^2}{\rho_3},
$$

$$
\beta = 2(\rho_1 - 1) \log \rho_1 + \frac{(1 - \rho_1)^2}{\rho_1} + (\rho_1 - 1)^2,
$$

$$
\gamma = (\rho_1 - 1) \log \rho_3 + (\rho_3 - 1) \log \rho_1 + 2(\rho_3 - 1)(\rho_1 - 1).
$$
If $\gamma^2 \leq 4\alpha\beta$, then this quadratic form in $a$ and $c$ is always non-negative. Expanding both sides, we need to check if
\[
(p_1 - 1)^2(\log \rho_3)^2 + (p_3 - 1)^2(\log \rho_1)^2 + 4(p_3 - 1)^2(p_1 - 1)^2 + 2(p_1 - 1)(p_3 - 1)\log \rho_3 \log \rho_1 + 4(p_1 - 1)^2(p_3 - 1)\log \rho_3 + 4(p_3 - 1)^2(p_1 - 1)\log \rho_1 \\
\leq \\
16(p_3 - 1)(p_1 - 1)\log \rho_1 \log \rho_3 + 8(p_3 - 1)(1 - p_1)^2 \rho_1 \log \rho_3 \\
+ 8(p_3 - 1)(p_1 - 1)^2\log \rho_3 + 8(p_3 - 1)^2(p_1 - 1)\log \rho_1 \\
+ 4(p_3 - 1)^2(1 - p_1)^2 \rho_1 + 4(p_3 - 1)^2(p_1 - 1)^2 \\
+ 8(1 - p_3)^2(p_1 - 1)\log \rho_1 + 4(1 - p_3)^2(1 - p_1)^2 \rho_3 \\
+ 4(p_1 - 1)^2(1 - p_3)^2 \rho_3.
\]
Notice that except for the first two terms on the LHS, all other terms are present on the RHS and in larger quantities. We bound these two terms as follows, in order to establish the inequality above:
\[
(p_1 - 1)^2(\log \rho_3)^2 \leq \frac{(p_1 - 1)^2(p_3 - 1)^2}{p_1},
\]
\[
(p_3 - 1)^2(\log \rho_1)^2 \leq \frac{(p_1 - 1)^2(p_3 - 1)^2}{p_1},
\]
where the first inequality follows because $|\log \rho_3| \leq (p_3 - 1)$ and from $p_1 \leq 1$, while the second inequality follows from $(\log x)^2 \leq (x - 1)^2$ for $x \in (0, 1)$ (the proof of this inequality is standard).

C. Counterexample for symmetric Markov chains

The results from the preceding sub-section give rise to an interesting conjecture: does every symmetric Markov chain exhibit convex relative entropy decay? The Markov chains encountered in section III, being symmetric, add further credence to this conjecture. However, we’ll show that this conjecture is not true by constructing a counterexample.

**Lemma 1.** Consider 3 states $a$, $b$, and $c$ in a reversible Markov chain. Let $\gamma_1$ and $\gamma_2$ be two allowed moves, with $\gamma_1(b) = a$ and $\gamma_2(b) = c$, with the jump rates $c(b, \gamma_1)$ and $c(b, \gamma_2)$ respectively. We claim that if $p_a < p_b < p_c$, then there exist $c(b, \gamma_1), c(b, \gamma_2) > 0$ such that the following holds–
\[
\sum_{\delta_1, \delta_2 \in \{\gamma_1, \gamma_2\}} c(b, \delta_1)c(b, \delta_2) \left[ \nabla_{\delta_1} \rho(b) \nabla_{\delta_2} \log \rho(b) + \frac{\nabla_{\delta_1} \rho(b) \nabla_{\delta_2} \rho(b)}{\rho(b)} \right] < 0.
\] 

**Proof:** We divide the expression in (13) by $c(b, \gamma_1)^2$ and think of it as a quadratic in $x = \frac{c(b, \gamma_2)}{c(b, \gamma_1)}$, say $px^2 + qx + r$. This quadratic is negative at some $x$ if it has real roots and $x$ lies between these roots. These roots, if real, cannot be negative because the assumption $p_a < p_b < p_c$ leads to $p, r > 0$ and $q < 0$. So if $q^2 - 4pr > 0$, then the quadratic has positive real roots and evaluates to negative values between these roots. The inequality $q^2 > 4pr$ upon expanding and canceling common terms reduces to
\[
2(p_a - p_b)(c(b, \gamma_1) - p_b) \log \frac{p_a}{p_b} > 2(p_a - p_b)(c(b, \gamma_2) - p_b) \log \frac{p_a}{p_b},
\]
which is true with weak inequality by the AM-GM inequality, and can be made to be true by appropriate choice of the density.

**Remark 3.** Expression [13] is essentially the contribution to the double derivative by state $b$, via the allowed moves $\gamma_1$ and $\gamma_2$.

Consider now the “symmetric star” Markov chain as in Figure 1. We choose $\rho_1 = \rho_2 = \ldots = \rho_n < \rho_0 < \rho_{n+1} = \cdots = \rho_{2n}$. We also choose the rates between $\{1, 2, \ldots, n\}$ and 0 to be all equal, and the rates between $\{n+1, n+2, \ldots, 2n\}$ and 0 to be all equal. These rates are chosen by Lemma 1 such that the contribution to the double derivative by 0 via the moves 0 → 1 and 0 → $n+1$ is negative, and equals $-C$. Let the contribution of $1, 2, \ldots, n$ be $A$ each, and that of $n+1, n+2, \ldots, 2n$ be $B$ each. The total contribution of 0 to the double derivative can be evaluated to be $-n^2C$, making the double derivative $n^2(-C) + nA + nB$. Clearly, for large enough $n$ this can be made as negative as desired.

V. RELATIVE ENTROPY DECAY IN JACKSON NETWORKS

Having looked at finite state chains, we now shift our focus to countable state Markov chains. An important class is those that arise from the queueing systems called Jackson networks. These are amongst the most well studied countable state Markov chains in networking applications. We first consider the simplest such Markov chain: the M/M/1 queue with an arrival rate $\lambda$ and a service rate $\mu$.

**Theorem 2.** Every stable M/M/1 queue has convex relative entropy decay.

**Proof:** Let the set of allowed moves be $\Gamma = \{+, -\}$, where $+(n) = n + 1$ and $-(n) = (n - 1)1_{n>0}$. The jump rates are given by
\[
c(n, +) = \lambda \text{ for all } n,
\]
\[
c(n, -) = \begin{cases} \mu & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}
\]
This Markov chain is reversible. Consider the terms in the summation on the RHS of equation (9). For $\eta = 0$, the only relevant choice is $(\eta, \gamma, \delta) = (0, +, +)$, since the other three choices give 0. For $\eta = 1$, let us exclude the choice

![Symmetric star Markov chain](image-url)
$(\eta, \gamma, \delta) = (1, -, -)$. Let us carry out the sum over all the remaining terms (i.e. other than the one corresponding to $(1, -, -)$). These terms can be partitioned into groups of four terms each, of the form

$$(\eta, \gamma, \delta) \in \{(n, +, +), (n + 1, +, -), (n + 1, - , +), (n + 2, - , -)\}, \quad n \geq 0.$$ 

Since we have

$$\pi(n)\lambda^2 = \pi(n + 1)\lambda \mu + \pi(n + 2)\mu^2, \quad n \geq 0,$$

it is natural to group each such set of four terms together. The contribution to the overall sum (with the term for $(1, -, -)$ excluded) of the sum of the $n$-th group of four terms is the product of $\pi(n)$ $\lambda^2$ with the sum of four other terms, each of which is of the form $\alpha \log \alpha - \alpha \log \beta + \beta - \alpha$ for some $\alpha, \beta > 0$, and is therefore nonnegative. The upshot is that the overall sum on the RHS of equation (9) for this example is bounded below by the term in the sum corresponding to the choice $(\eta, \gamma, \delta) = (1, -, -)$. But this contribution, i.e.

$$\mu^2 \left( \nabla_+ \rho(1) \nabla_+ \rho(1) + \nabla_- \rho(1)^2 \rho(1) \right),$$

is also nonnegative. This concludes the proof.

An important generalization of the M/M/1 queue is an open Jackson network [8]. An open Jackson network with $n$ nodes is one where the arrival of jobs is memoryless, at some rate $\alpha > 0$. Each arrival is independently routed to node $j$ with probability $p_{0j} \geq 0$, where $\sum_{j=1}^{n} p_{0j} = 1$. When a job is served at node $i$, it can either go to some other node $j$ with probability $p_{ij}$ or exit the network with probability $p_{i0} = 1 - \sum_{j=1}^{n} p_{ij}$. Hence the overall arrival rate at node $i$, which we call $\lambda_i$, satisfies

$$\lambda_i = \alpha p_{0i} + \sum_{j=1}^{n} \lambda_j p_{ji}, \quad i = 1, 2, ..., n.$$  \hspace{1cm} (14)

Define $a = \alpha [p_{01}, p_{02}, ..., p_{0n}]^T$. We can write $\lambda = (I - P^T)^{-1}a$, where $P$ is the routing matrix $\{p_{ij}\}$, which we assume has its Perron-Frobenius eigenvalue strictly less than 1. Here $\lambda$ denotes the column vector of the $\lambda_i$. Let the service rate at node $i$ be $\mu_i$. The service time of each job at each node is assumed to be exponentially distributed with this rate, independently over each visit of each job to each node, and independent of the arrivals and the routing. A fundamental result for such a network is that it is stable if and only if $\lambda_i < \mu_i$ for all $i$. In addition, such a network is quasi-reversible [8], which implies that the stationary distribution $\pi(x_1, x_2, ..., x_n)$ has a product form given by

$$\pi(x_1, x_2, ..., x_n) \propto \prod_{i=1}^{n} \left( \frac{\lambda_i}{\mu_i} \right)^{x_i}.$$ \hspace{1cm} (15)

As we saw in Claim 2, every stable M/M/1 queue is reversible and has convex relative entropy decay. We ask the following question: does every stable Jackson network also have convex relative entropy decay?

**Theorem 3.** The only Jackson networks having convex relative entropy decay for every choice of service rates that ensure stability are those whose routing matrices are diagonal.

**Proof:** Consider a stable Jackson network with $n$ nodes and a routing matrix $P$. Since the Markov chain resulting from such a Jackson network is not always reversible, we need to use equation (8) which uses both the forward and the reverse Markov chains. Let $e_1, e_2, ..., e_n$ denote the standard basis vectors, and let $e_{ij} := e_j - e_i$. Given a state $x = (x_1, x_2, ..., x_n)$ in the forward Markov chain, there are three types of jumps possible:

- **Transition:** For any distinct $i$ and $j$ with $x_i > 0$, jumps of the form $x \rightarrow x + e_{ij}$. These take place at a rate $\mu_i p_{ij}$.
- **Departure:** For any $i$ with $x_i > 0$, jumps of the form $x \rightarrow x - e_i$. These take place at a rate $\mu_i p_{i0}$.
- **Arrival:** For any $i$, jumps of the form $x \rightarrow x + e_i$. These take place at a rate $\alpha p_{0i}$.

For the time reversed network, the possible jumps and the rates are given by:

- **Transition:** For any distinct $i$ and $j$ with $x_i > 0$, jumps of the form $x \rightarrow x + e_{ij}$. These take place at a rate $(p_{ij} \lambda_i) / \mu_i$.
- **Departure:** For any $i$ with $x_i > 0$, jumps of the form $x \rightarrow x - e_i$. These take place at a rate $(\alpha p_{0i} \lambda_i) / \mu_i$.
- **Arrival:** For any $i$, jumps of the form $x \rightarrow x + e_i$. These take place at a rate $\alpha p_{0i}$.

Note that in both the forward and the reversed chains, transition and departure jump rates are proportional to $\mu_i$ whereas the arrival rate is not dependent on $\mu_i$. Since both the forward and reversed Markov chains are completely described, we can use (8) to explicitly compute the contribution of each state $x$ to the double derivative expression. Let us call a state $x$ to be at level $N$ if $x_1 + x_2 + ... + x_n = N$. We first fix an initial density $\rho$ such that $\rho(x) = 1$ for all states $x$ at level 2 and higher. We'll now examine the contribution to the double derivative from states lying in different levels.

**Levels 3 and higher:**

Every state in level 3 or higher, has jumps going either in the same level, or at the levels just above and below it. Since $\rho$ is 1 in all these levels, it's easy to see that $L^* \rho(x) = 0$ for $x$ in these levels, and thus so is their contribution to the double derivative.

**Level 1:**

Denote states in level 1 by the standard basis vectors $e_1, e_2, ..., e_n$. From (13) we now examine the contribution from $e_i$ to the double derivative. Observe that if we pull out the $\pi(e_i)$ term from the contribution of $e_i$, the remaining expression is a quadratic in $\mu_i$. Since $\pi(e_i) \propto \lambda_i / \mu_i$, the combined contribution from all the $e_i$’s is of the form

$$\sum_i a_i \mu_i + b_i + c_i / \mu_i,$$

for some $a_i, b_i, \text{ and } c_i$.

Now suppose we’re able to make $a_i$ negative for a particular $i$, say $i^*$. In such a case, we can make $\mu_{i^*}$ large enough so that (16) can be made negative. We’ll now show that such an $i^*$ always exists, so long as the routing matrix is not diagonal. Consider the structure in Figure 2 in the forward Markov chain. It is not hard to see that such a structure has to exist for some $i$, if the routing matrix is not diagonal– one merely has to
look at “exit” nodes $i$, i.e., $i$ such that $p_{i0} > 0$, and choose a node $j$ that feeds into it. We can explicitly write down the contribution of node $i^*$ using the two expressions

$$
(\mathcal{L}^* \rho)(e_{i^*}) = \sum_{j \neq i^*} (\rho(e_j) - \rho(e_{i^*})) \left( p_{j,i^*} \frac{\lambda_j}{\lambda_{i^*}} \right) \mu_{i^*} + \\
\sum_j (\rho(e_j + e_{i^*} - \rho(e_{i^*})) \lambda_j p_{j0} + (\rho(0) - \rho(e_{i^*})) \left( \alpha p_{0i^*} \right) \mu_{i^*}, \\
(\mathcal{L} \log \rho)(e_{i^*}) = \sum_{j \neq i^*} \log \frac{\rho(e_j)}{\rho(e_{i^*})} \mu_{i^*} \mu_{j,i^*} \cdot \\
\sum_j \log \frac{\rho(e_j + e_{i^*})}{\rho(e_{i^*})} \alpha_{p0j} \cdot \\
+ \log \frac{\rho(0)}{\rho(e_{i^*})} \mu_{i^*} \mu_{i^*0} .
$$

We now choose $\rho(e_j) = 1$ for all $j \neq i^*$, and fix the order relation $1 > \rho(e_{i^*}) > \rho(0)$. (As mentioned earlier, we are free to scale the density at will when studying the sign of the second derivative of the relative entropy, so we do not worry about the normalization condition.) In equation (17), the first and the third term comprise the linear term in $\mu_{i^*}$. We fix $\rho(e_{i^*})$ small enough such that this linear term is strictly positive, independent of $\rho(0)$ (as long as $\rho(0) < \rho(e_{i^*})$). Similarly, in equation (18) the first and third term comprise the linear term in $\mu_{i^*}$. Note that as $\rho(0) \to 0$, the coefficient of this linear term tends to negative infinity. Thus, the coefficient of $\mu_{i^*}^2$ in $(\mathcal{L}^* \rho)(e_{i^*})$ can be made strictly negative for a small enough $\rho(0)$. Further, the coefficient of $\mu_{i^*}^2$ in $((\mathcal{L}^* \rho)(e_{i^*}))^2/\rho(e_{i^*})$ is bounded as $\rho(0) \to 0$. We can thus convince ourselves that for a suitably small $\rho(0)$, $\alpha_{i^*}$ as defined in (16) can be made strictly negative, and thus conclude that it is possible to fix $\rho$ and $\mu_1, \mu_2, ..., \mu_n$ such that the combined contribution of level 1 states to the double derivative is negative. Note that we can choose all the $\mu_i$ to $K \mu_{i^*}$, for some large $K$ and make this contribution even more negative. Our motivation in doing so is to make the contribution from level 1 a large enough negative number that it dwarfs the contribution from all other levels. But before that, we need to check how scaling by $K$ affects contributions from other levels. Note that levels 3 and higher continue to contribute 0.

**Level 2:**

As $K$ increases, level 2 contributions remain bounded. This is because for level 2, the stationary distribution as a function of $K$ scales like $\pi \propto \frac{1}{K}$, and the double derivative contribution for a state with $\pi$ pulled out is quadratic in $K$.

**Level 0:**

Since the stationary distribution at level 0 is independent of $\mu_i$‘s, and all jumps from level 0 have rates independent of $\mu_i$‘s, the contribution of level 0 does not scale with $K$.

The total double derivative, which is a sum of the contributions from all the levels, can thus be made negative for a large enough $K$.

Instead of an open Jackson network, one might also be interested in a closed Jackson network where a finite number of jobs are routed through queues, with no jobs entering or leaving the system. The Markov chain generated is a finite state Markov chain. Just like the M/M/1 queue, the simplest closed Jackson network with 2 queues has convex relative entropy decay. Using a similar strategy as the one used for open networks, we can show that for closed networks with more than 2 queues, convex relative entropy decay does not always hold.

**VI. Conclusion**

We looked at various Markov chains of interest and examined convexity of relative entropy decay. Simulations suggest that choosing an arbitrary rate matrix and an initial distribution $\mu$, and evaluating $\frac{d^2}{dt^2} h(\mu_t | \pi)$, invariably leads to a positive value of the double derivative; thus counterexamples are not easy to find and require careful construction. Choosing notation appropriately, we developed techniques to generate a rich collection of counterexamples. We looked at general reversible finite state Markov chains, and even with stringent conditions like a symmetric rate matrix, convexity of relative entropy decay cannot be guaranteed. The property of convexity of relative entropy decay for the M/M/1 queue does not extend to Markov chains arising out of nontrivial Jackson networks, even though these networks have very similar properties to a network of independent M/M/1 queues.

**Acknowledgements**

The research of the authors was supported by the ARO MURI grant W911NF-08-1-0233, Tools for the Analysis and Design of Complex Multi-Scale Networks, the NSF grant CNS-0910702, and the NSF Science & Technology Center grant CCF-0939370, Science of Information.

**References**


