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THE GREATEST OF A FINITE SET OF RANDOM VARIABLES

Charles E. Clark

System Development Corporation, Santa Monica, California

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The variables $\xi_1, \cdots, \xi_n$ have a joint normal distribution. We are concerned with the calculation or approximation of $\max(\xi_1, \cdots, \xi_n)$. Current analyses and tables handle the case in which the ξ_i are independently distributed with common expected values and common variances. This paper presents formulas and tables for the most general case with $n=2$. When $n>2$, the problem becomes cumbersome. This paper presents formulas and tables that permit approximations to the moments in case $n>2$. The moments are approximated by iteration of a three-parameter computation or, alternatively, through successive use of a three-parameter table, which is given. Recent applications of the theory are described.

THE GREATEST of two or more random variables enters many operations-research analyses. For example, the number of sorties available at a SAC base is the greatest of the number of bombers available, the number of crews available, and other random variables such as refueling capability. Two further illustrations are described below in the introductory paragraphs of sections 4 and 5; these paragraphs can be read at this point because they do not use the notation developed in the intervening discussion.

If $\xi_1, \cdots, \xi_n$ are independently and normally distributed with a common expected value and variance, the maximum of the n variables is simply the extreme value of a sample of size n from a normal distribution. Hence the literature on order statistics gives considerable information about the maximum. This literature is outlined in the bibliography listed in references 3 and 4. Numerical tables of moments of this maximum appear in references 2 and 4. If one considers this same maximum when the distribution is not normal, analytic results concerning the moments are given in reference 5 and in the bibliography listed in that paper. However numerical tables are not available for the nonnormal case.

In problems of operations research one often considers independently distributed variables. Moreover it is sometimes legitimate to assume normality. But in many problems the variables do not have a common expected value, variances are unequal, or correlation exists. In such cases available theory and numerical tables are not available. This gap is filled by the present paper.

This paper approximates the first four moments of $\max(\xi_1, \cdots, \xi_n)$ where the ξ's have a joint normal distribution. This joint normal distribution is unrestricted; the expected values, variances, and correlations are arbitrary. Applications to nonnormal distributions are discussed. The analytic results are stated in the section following this introduction. Discussions of two important recent applications follow. The errors in the approximations are discussed. A numerical table is presented, and the discussion terminates with the mathematical derivations.

1. THE ANALYTIC RESULTS

LET ξ, η, and τ be normally distributed with expected values μ_1, μ_2, and $E(\tau)$, respectively, and with variances σ_1^2, σ_2^2, and $V(\tau)$. The expected value and variance of τ are not specified because all results to be obtained involve τ only in statements concerning correlations, and the results are independent of the expected value and variance of τ. If r denotes the coefficient of linear correlation, we write $r(\xi, \eta)=\rho$, $r(\xi, \tau)=\rho_1$, and $r(\eta, \tau)=\rho_2$. We shall use the notation $\varphi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$ and $\Phi(x)=\int_{-\infty}^{x}\varphi(t)\,dt$.

If
$$\sigma_1-\sigma_2=\rho-1=0, \tag{1}$$
ξ and η differ by a constant. The analysis developed below does not apply under this restriction. However, much simpler computations will handle the special case. These computations are omitted. In Table III below, the special case is included.

Let ν_i be the ith moment (about zero) of the random variable $\max(\xi, \eta)$. We shall use the notation
$$a^2=\sigma_1^2+\sigma_2^2-2\,\sigma_1\,\sigma_2\,\rho.$$

This expression is positive because we assume that (1) does not hold. Introducing the notation
$$\alpha=(\mu_1-\mu_2)/a,$$
we can prove (*see* Sec. 7) that
$$\nu_1=\mu_1\,\Phi(\alpha)+\mu_2\,\Phi(-\alpha)+a\,\varphi(\alpha), \tag{2}$$

$$\nu_2 = (\mu_1^2+\sigma_1^2)\,\Phi(\alpha)+(\mu_2^2+\sigma_2^2)\,\Phi(-\alpha)+(\mu_1+\mu_2)\,a\,\varphi(\alpha), \tag{3}$$

$$\begin{aligned}\nu_3 = {} & (\mu_1^3+3\,\mu_1\,\sigma_1^2)\,\Phi(\alpha)+(\mu_2^3+3\,\mu_2\,\sigma_2^2)\,\Phi(-\alpha) \\ & +[(\mu_1^2+\mu_1\,\mu_2+\mu_2^2)\,a+(2\,\sigma_1^4+\sigma_1^2\,\sigma_2^2+2\,\sigma_2^4 \\ & -2\,\sigma_1^3\,\sigma_2\,\rho-2\,\sigma_1\,\sigma_2^3\,\rho-\sigma_1^2\sigma_2^2\,\rho^2)\,a^{-1}]\,\varphi(\alpha),\end{aligned} \tag{4}$$

$$\begin{aligned}\nu_4 = {} & (\mu_1^4+6\,\mu_1^2\,\sigma_1^2+3\,\sigma_1^4)\,\Phi(\alpha)+(\mu_2^4+6\,\mu_2^2\,\sigma_2^2+3\,\sigma_2^4)\,\Phi(-\alpha) \\ & +\{(\mu_1^3+\mu_1^2\,\mu_2+\mu_1\,\mu_2^2+\mu_2^3)\,a-3\,\alpha(\sigma_1^4-\sigma_2^4) \\ & +4\,\mu_1\,\sigma_1^3\,[3\,(\sigma_1-\sigma_2\,\rho)/a-(\sigma_1-\sigma_2\,\rho)^3/a^3] \\ & +4\,\mu_2\,\sigma_2^3\,[3\,(\sigma_2-\sigma_1\,\rho)/a-(\sigma_2-\sigma_1\,\rho)^3/a^3]\}\,\varphi(\alpha),\end{aligned} \tag{5}$$

$$r[\tau, \max(\xi, \eta)] = [\sigma_1\,\rho_1\,\Phi(\alpha)+\sigma_2\,\rho_2\,\Phi(-\alpha)]/(\nu_2-\nu_1^2)^{1/2}. \tag{6}$$

The formulas for the ν_i permit calculations related to the greater of two normal variables. Formula (6) is used in estimating moments of the greatest of more than two normally distributed variables. This fact will be illustrated next.

2. THE EXPECTED VALUE AND VARIANCE OF THE GREATEST OF A FINITE SET OF NORMALLY DISTRIBUTED VARIABLES

WE SHALL illustrate the utility of the formulas of Sec. 2, especially formula (6). Suppose that we know the expected values, variances, and coefficients of linear correlation for four normally distributed variables ξ, η, ζ, and ω. We wish to estimate the first four moments of $\max(\xi, \eta, \zeta, \omega)$. We consider

$$\max(\xi, \eta, \zeta) = \max[\max(\xi, \eta), \zeta]. \tag{7}$$

If $\max(\xi, \eta)$ were normally distributed, we could calculate the moments of (7) in the following manner. Let E and V denote expected value and variance, respectively. To calculate the first four moments of (7) with use of (2) through (5), we need $\mathrm{E}[\max(\xi, \eta)]$, which is obtained by use of (2), $\mathrm{V}[\max(\xi, \eta)]$, which is obtained from (3), $\mathrm{E}(\zeta)$, which is given, $\mathrm{V}(\zeta)$, which is given, and $r[\max(\xi, \eta), \zeta]$. This coefficient of correlation can be obtained from (6). With these results, one would use (2) through (5) to calculate the first four moments of (7).

This last calculation would be inaccurate because $\max(\xi, \eta)$ is not normally distributed. However, we shall accept the results of the calculation as approximations. The errors in the approximations will be discussed below in Sec. 5.

Next we consider

$$\max(\xi, \eta, \zeta, \omega) = \max[\max(\xi, \eta, \zeta), \omega]. \tag{8}$$

To obtain the expected value of this last variable by use of (2), we need

$$r[\max(\xi, \eta, \zeta), \omega] = r\{\max[\max(\xi, \eta), \zeta], \omega\}. \tag{9}$$

To estimate the right side of this equation by use of (6) we need $r(\omega, \zeta)$, which is given, and $r[\omega, \max(\xi, \eta)]$, which is approximated as was $r[\zeta, \max(\xi, \eta)]$ in the analysis above. Thus (9) can be approximated. Hence, we can get approximations to the moments of (8) with use of (9) together with the expected value and variance of (7) as calculated above. The errors of approximation result from applying formulas for normally distributed variables to $\max(\xi, \eta)$, $\max(\xi, \eta, \zeta)$, etc.

Clearly, this approximate calculation can be extended to any finite number of normal variables. The use of such approximations in operations analysis is illustrated in the next two sections.

3. APPLICATION TO AN ASSEMBLY-LINE PROBLEM

THE RESULTS of this paper have found application to the following problem. An assembly line has n stations through which each fabricated article must pass in succession. As soon as all n stations complete their tasks, the $n-1$ partially fabricated articles are moved to more advanced stations, and a new article is started at the first station. The problem is to estimate the expected value and standard deviation of the time between shifts. This is the expected value and standard deviation of the greatest of n random times, and normality assumptions lead us to apply the results of this paper.

The n times would be correlated if manpower would be transferred from a station ahead of schedule to a station behind schedule. In the real problem studied by the author and his colleagues, it was legitimate to assume zero correlation.

4. A SECOND APPLICATION: PARTIALLY ORDERED SETS OF ACTIVITIES

THE NETWORK of Fig. 1 represents a partially ordered set of activities. The time required for activity a is uncertain, being normally distributed with expected value $\mathrm{E}(a)$ and variance $\mathrm{V}(a)$. Similar notation is used for the other activities b through e. The five activity times are independent. Activities a and b start at time zero. As soon as b is completed at time B, both c and e commence. As soon as both a and c are completed at time C, activity d commences. The problem is to estimate the expected value and variance of D, the time at which both d and e are completed. The time of D is the greater of two times, the time of C plus the time of d, and the time of B plus the time of e. An important feature of the problem is that the times of B and C are correlated. Indeed, both of these times are influenced by the time of b. The analysis of the correlation will require formula (6) as we shall show below.

There are two computational procedures that might be followed in this problem. As a matter of notation, if x denotes an activity, the same symbol x denotes the time of the activity. The time of D is the greatest of the three random variables $a+d$, $b+c+d$, and $b+e$. The correlations among these three sums could be calculated by the procedures described below in this section. Since the expected values and variances of the three sums are easily calculated, we would have the requisite information for the estimation of the expected value and variance of $D = \max(a+d, b+c+d, b+e)$; we could compute as in Sec. 2.

However, this calculation requires a study of all the 'paths' from start to finish in the network of activities. If there were thousands of activities, this number of paths could be intractably large. Hence, we are led to use a

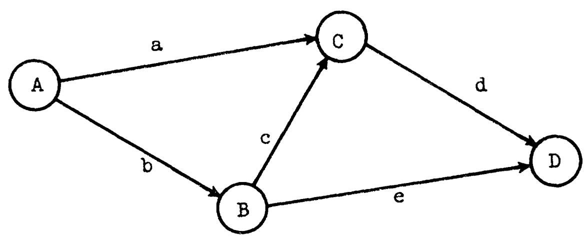

Figure 1

different calculation, which may not be so efficient for the present problem, but which is better for large problems. The combinatorial magnitude of the following analysis is at worst that of the number of pairs of events, and in many cases this is much smaller than the number of paths.

The alternative procedure is the following: The nodes A, B, C, and D of Fig. 1 are called events. The symbol for an event will also denote the time of the event. In the course of the calculation we shall estimate the expected value and variance of the time of each event. Furthermore, the coefficient of correlation will be estimated for some pairs of event times. The required pairs will be indicated later.

The exposition of this section will be more ponderous than required for the analysis of Fig. 1. Our objective is to indicate that a network with greater detail can be analyzed by the same procedure.

The first step is to arrange the events in a linear order such that in the linear order each event is preceded by all the events that must occur prior to it. In the case of Fig. 1, there is only one such ordering, namely,

$$A, B, C, D. \tag{10}$$

For example, event C could not occur until both A and B have occurred;

hence, A and B must precede C in the linear ordering. In more complex networks the linear ordering is not uniquely determined, and the construction of a specific linear ordering requires some manipulation.*

The expected value and variance of each event time will be estimated, the events being studied one-by-one in the order in which they appear in the linear ordering (10). As we analyze an event we must consider all its immediately preceding events. The events immediately preceding D in Fig. 1 are B and C. In general, an event X is an immediate predecessor of an event Y if the geometric model of the network contains an arrow leading from X to Y. In Fig. 1 the event A precedes D but is not an immediate predecessor. We shall see below that before estimating the expected value and variance of an event time, we must estimate the coefficients of correlation between the times of each pair of immediately preceding events.

The first event in the linear ordering (10) is A. Clearly, $\mathrm{E}(A)=\mathrm{V}(A)=0$. The next event in the linear ordering is B. We must select all the immediately preceding events, and these include the single event A. Since B has only one immediate predecessor, we see directly that $\mathrm{E}(B)=\mathrm{E}(A)+\mathrm{E}(b)=\mathrm{E}(b)$, $\mathrm{V}(B)=\mathrm{V}(A)+\mathrm{V}(b)=\mathrm{V}(b)$.

The next event in the linear ordering (10) is C. We recognize that the time of C is $\max(a, B+c)$. To estimate the expected value and variance of C by the method of this paper, we need $\mathrm{E}(a)$, which is given, $\mathrm{V}(a)$, which is given, $\mathrm{E}(B+c)=\mathrm{E}(B)+\mathrm{E}(c)$, which is known because $\mathrm{E}(B)$ was computed above and $\mathrm{E}(c)$ is given, $\mathrm{V}(B+c)=\mathrm{V}(B)+\mathrm{V}(c)$, which is known because $\mathrm{V}(B)$ was computed above and $\mathrm{V}(c)$ is given, and $r(a, B+c)$, which is zero because a is independent of $B+c$. Since all necessary information is available, we can estimate $\mathrm{E}(C)$ and $\mathrm{V}(C)$.

The next event in the linear ordering (10) is D. The time of D is $\max(B+e, C+d)$. We shall estimate the expected value and variance of D by regarding it as the greater of two random variables. We estimate the expected values and variances of $B+e$ and $C+d$ in the manner illustrated above. In addition we shall estimate $r(B+e,C+d)$, but this requires some statistical analysis which is developed next.

We digress momentarily to prove an elementary relation involving correlations. Let $\mathrm{C}(P, Q)$ denote the covariance of any pair of random variables P and Q. Given four random variables, X, Y, A, and B, such that each of A and B is independent of the other three we shall prove that

$$r(X+A.Y+B)=\mathrm{V}^{1/2}(X)\,\mathrm{V}^{1/2}(Y)\,r(X,Y)/\mathrm{V}^{1/2}(X+A)\,\mathrm{V}^{1/2}(Y+B). \quad (11)$$

We shall use the fact that the coefficient of linear correlation between two random variables is the covariance divided by the product of the standard deviations. Our first use of this fact enables us to write

$$r(X+A,\, Y+B)=\mathrm{C}(X+A,\, Y+B)/\mathrm{V}^{1/2}(X+A)\,\mathrm{V}^{1/2}(Y+B). \quad (12)$$

* This ordering problem is solved in reference 1.

The definition of the covariance implies that

$$C(X+A, Y+B) = C(X, Y) + C(X, B) + C(A, Y) + C(A, B) = C(X, Y),$$

the three covariances dropping out because of the hypothesis of independence. Substitution reduces (12) to (11).

We return to the estimation of $r(B+e, C+d)$. Since e and d are independent of each other and of both B and C, formula (11) is applicable. To complete the estimation of $r(B+e, C+d)$ by formula (11), we need only determine $r(B, C)$ because all other required factors are readily determined.

The estimation of $r(B, C) = r[b, \max(a, b+c)]$ proceeds as follows. We shall use (6) with τ replaced by b and with ξ and η replaced by a and $b+c$. As a preliminary step let us estimate using (11) that

$$r(b, b+c) = V^{1/2}(b)\ V^{1/2}(b)\ r(b,b)/V^{1/2}(b)\ V^{1/2}(b+c) = V^{1/2}(b)/V^{1/2}(b+c).$$

The numerator and denominator of this last fraction are readily calculated. On the other hand, clearly $r(b, a)$ and $r(a, b+c)$ are zero because of independence. With these results we recognize that for the three random variables a, $b+c$, and b, we know all three expected values and variances, and we know all three coefficients of correlation. Hence, as indicated in Sec. 3, we can use (6) to estimate $r[b, \max(a, b+c)]$. But this is $r(B, C)$.

The reader will recognize that we have outlined a computational procedure that produces estimates of the first four moments of D.

5. ACCURACY OF THE NORMAL APPROXIMATION TO NONNORMAL VARIABLES

IT HAS been suggested that the results obtained above for normal distributions can be used with adequate accuracy in some cases involving nonnormal distributions. The present section illustrates this fact numerically.

Suppose that we wish to approximate the moments of $\max(\xi_1, \xi_2)$ where ξ_1 and ξ_2 are not normally distributed. Let η_i, $i=1, 2$, be normally distributed with the same expected value and variance as ξ_i. We shall show that in many cases the moments of $\max(\xi_1, \xi_2)$ are adequately approximated by the moments of $\max(\eta_1, \eta_2)$.

If the difference $E(\xi_1) - E(\xi_2)$ is large relative to the greater of $V^{1/2}(\xi_1)$ and $V^{1/2}(\xi_2)$, the random variable $\max(\xi_1, \xi_2)$ is practically identical with ξ_1. In such a case, no computations are required in order to approximate the first two moments of $\max(\xi_1, \xi_2)$. Certainly there would be no significant error involved in replacing ξ_1 and ξ_2 by normal approximations (this would not be true if one were to consider moments of higher order than the second).

However, if $E(\xi_1) - E(\xi_2)$ is small relative to the standard deviations, $\max(\xi_1, \xi_2)$ differs considerably from both ξ_1 and ξ_2. In this case, the use of normal approximations for ξ_1 and ξ_2 could conceivably produce serious errors in the approximation of the expected value and variance of the greater of the two variables. Intuitively, the most serious situation is that in which the expected values are equal. This fact, in part, motivates the consideration of the following numerical illustrations.

Consider two variables each of which is distributed as the greatest of 500 standard normal variables. It is shown in reference 2 that the expected value and standard deviation of each of these variables are 3.03670 and 0.3704, respectively. These variables are not normally distributed. Indeed, their skewness is 0.570 and their kurtosis is 1.003; these numbers are given in reference 2. Approximating these two variables by normal variables with the same expected value and variance, we can apply (2) to approximate the expected value of the greater as 3.2457. The correct value is given in reference 2 as 3.24144.

TABLE I
THE EXPECTED VALUE OF THE GREATEST OF n STANDARD NORMAL VARIABLES

n	$E[\max(\xi_1, \cdots, \xi_n)]$	Approximation
2	0.5642	0.5642
3	0.8463	0.8476
4	1.0294	1.0310
5	1.1630	1.1643
6	1.2672	1.2679
7	1.3522	1.3522
8	1.4236	1.4230
9	1.4850	1.4837
10	1.5388	1.5367

TABLE II
THE EXPECTED VALUE OF THE GREATEST OF n VARIABLES WITH EXPECTED VALUE 0 AND VARIANCE 1

n	Exponentially distributed	Uniformly distributed
2	0.5000	0.5774
3	0.8333	0.8660
4	1.0833	1.0392
5	1.2833	1.1547
6	1.4500	1.2372
7	1.5929	1.2990
8	1.7179	1.3472
9	1.8290	1.3856
10	1.9290	1.4171

As a second illustration, consider the greatest of n standard normal variables $\xi_1, \xi_2, \cdots, \xi_n$. The middle column of Table I is the expected value of $\max(\xi_1, \cdots, \xi_n)$. These numbers are obtained from reference 2. We can approximate these expected values in the following manner. For $n=2$, the expected value is given exactly by (2). Furthermore, the variance of the greater of two standard normal variables is given exactly by (3). Let η_2 be normally distributed with expected value $E[\max(\xi_1, \xi_2)]$ and variance $V[\max(\xi_1, \xi_2)]$. We use the moments of $\max(\eta_2, \xi_3)$ as approximations for those of $\max(\xi_1, \xi_2, \xi_3)$. Let η_3 be normally distributed with expected value and variance equal to the corresponding moments of $\max(\eta_2, \xi_3)$. We use the moments of $\max(\eta_3, \xi_4)$ as approximations for the moments of $\max(\xi_1, \cdots, \xi_4)$. In this way, one obtains the last column of Table I. There are few if any problems in operations analysis that require smaller errors of approximation.

The errors in the method of this paper result from the replacement of nonnormal distributions by normal approximations. Some intuitive feeling for the magnitudes of these errors can be obtained. Let us consider the distribution with probability density $e(x)=\exp[-(x+1)]$, $x \geqq -1$, and with probability density zero for $x<-1$. The expected value and variance of this distribution are 0 and 1 respectively. The third central moment divided by the cube of the standard deviation is $\gamma_1=2$. When one subtracts 3 from the fourth central moment divided by the fourth power of the standard deviation, one obtains $\gamma_2=6$. Since $\gamma_1=\gamma_2=0$ for normal distributions, these γ's measure the obvious fact that $e(x)$ is far from normal. If η_i, $i=1, \cdots, n$, are independently distributed with probability density $e(x)$, it is easy to calculate the moments of $\eta^{(n)}=\max(\eta_1, \cdots, \eta_n)$. Indeed, the distribution function of each η_i is $1-\exp[-(x+1)]$ for $x \geqq -1$, and the distribution function of $\eta^{(n)}$ is the nth power of the distribution function of η_i. In this way one obtains the second column of Table II. Let us next approximate these expected values of the $\eta^{(n)}$ by the method of this paper. Since the expected value and variance of each η_i are 0 and 1 respectively, the approximating normal distributions are those used in the calculations of Table I. Hence the numbers in the column 'Approximation' in Table I are the approximations by the method of this paper of the numbers in the second column of Table II. The errors of approximation can be observed by comparing the two tables. These errors of approximation range from $+13$ per cent to -20 per cent.

We recall that for a normal distribution $\gamma_1=\gamma_2=0$. In view of the numerical results concerning $e(x)$, it appears that the errors of approximation increase from nearly zero to roughly 20 per cent as γ_1 and γ_2 increase from $\gamma_1=\gamma_2=0$ to $\gamma_1=2$, $\gamma_2=6$. In the light of this statement we remark that it is unusual for the greater of two normal variables to have γ's exceeding 2 and 6 respectively; this will appear below in Table III.

To give an illustration involving negative γ_2 we consider the distribution with probability density $u(x)=(12)^{-1/2}$ for $-3^{1/2} \leqq x \leqq 3^{1/2}$, and $u(x)=0$ for $|x|>3^{1/2}$. The expected value and variance of this distribution are 0 and 1, and we have $\gamma_1=0$, $\gamma_2=-1.2$. If the independently distributed variables ζ_i, $i=1, \cdots, n$, have the probability density $u(x)$, the distribution function of $\max(\zeta_1, \cdots, \zeta_n)$ is easily obtained as the nth power of the quotient of $x+3^{1/2}$ divided by $2(3)^{1/2}$. Numerical values of max $(\zeta_1, \cdots, \zeta_n)$ are given in the last column of Table II. Comparison with the last column of Table I shows that there is serious but not huge error in using the normal approximation to a distribution with uniform probability density; the errors in the tabulated results range up to 8 per cent. We might say very roughly that to apply the approximations of this paper, one should have distributions with γ_2 greater than -1.

To terminate this discussion of error we can state only subjective attitudes. As far as the author is aware, in most applications the procedures of this paper constitute the only way of avoiding costly computations if one wishes to approximate the expected value of the greater of a set of random variables. The numerical illustrations suggest situations in which the error of approximation is great. However for many applications these numerical results have satisfied the author that, relative to accuracy and cost of computation, the procedure of this paper is the best available. In case of doubt concerning accuracy, the author would make an *ad hoc* appraisal of error in consideration of the specific distributions involved.

6. A NUMERICAL TABLE

TABLE III presents numerical results that can be obtained from the formulas of this paper.* The table applies when $\mu_1=0$, $\sigma_1=1$, and $\sigma_2\leqq 1$. There is no loss of generality in this restriction because a linear transformation can be applied to any pair of normally distributed variables with the result that the variables reduce to a pair satisfying this restriction. Indeed, for arbitrary μ_1, μ_2, σ_1, σ_2, and ρ, one can use the larger σ as the unit of measurement, and the μ corresponding to the larger σ can be transformed into zero; in case $\sigma_1=\sigma_2$, either μ can be transformed into zero.

In Table III, the parameters μ_2 and σ_2 are written μ and σ, respectively. The standard deviation of $\max(\xi, \eta)$ is written $\mu_2^{1/2}$ and is equal to $(\nu_2-\nu_1^2)^{1/2}$. Furthermore, γ_1 denotes the third central moment of $\max(\xi, \eta)$ divided by $\mu_2^{3/2}$, and γ_2 is -3 plus the fourth central moment divided by μ_2^2. Finally, the table uses the notation

$$A=\Phi(-\mu_2/a)/(\nu_2-\nu_1^2)^{1/2}, \qquad B=\sigma_2\,\Phi(\mu_2/a)/(\nu_2-\nu_1^2)^{1/2}, \qquad C=A\pm B,$$

C being defined only if $\rho=\pm 1$, and the ambiguous sign in the definition of C is taken as plus if $\rho=1$ and minus if $\rho=-1$.

The utility of A and B is apparent from (6), which implies that $r[\tau, \max(\xi, \eta)]=A\rho_1+B\rho_2$. If $\rho=\pm 1$, it follows that $\rho_1=\pm\rho_2$. Hence, if $\rho=\pm 1$, the last expression becomes $(A\pm B)\rho_1$, and $r[\tau, \max(\xi, \eta)]=C\rho_1$.

The table includes results for the limiting case in which $\sigma_2=0$. The analysis of this special case is relatively simple, and it is omitted.

The arguments of μ, σ, and ρ in the table were chosen as follows. Consider negative values of μ. If μ were less than -5, the second variable η would almost never be greater than ξ, and $\max(\xi, \eta)$ would have a distribu-

* Some of the numbers in Table III were obtained from a table prepared for internal use by Project PERT, Special Projects Office of the U. S. Navy, MR. WILLARD FAZAR, Director. This PERT table was constructed under the author's supervision as part of the study reported in reference 1.

tion almost identical with the distribution of ξ; the expected value ν_1 of $\max(\xi, \eta)$ would be slightly greater than 0, the expected value of ξ.

In Table III there are no entries with μ negative and $\nu_1 < 0.005$. However, all combinations of $\mu = -\infty(0.5)0$, $\sigma = 0(0.2)1$, and $\rho = -1(0.5)1$ are included for which $\nu_1 \geqq 0.005$ (except that ρ must be zero when σ is zero). Hence, a bound is known for ν_1 in cases of missing tabular arguments.

If μ is positive and large, ν_1 will be slightly greater than μ. For positive μ the tabular arguments are chosen so that corresponding to any missing set of arguments, the value of ν_1 is between μ and $\mu + 0.005$.

7. DERIVATIONS

The derivations of the analytic results are long and tedious. We shall indicate the course of these derivations and present milestones that turn up enroute. We continue to use notation introduced at the beginning of Sec. 1.

The probability density of ξ and η is

$$\varphi(x,y) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}.$$

We write

$$\nu_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\max(x, y)]^i \varphi(x,y)\, dx\, dy = \nu_{i1} + \nu_{i2},$$

where

$$\nu_{i1} = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} y^i \exp\left[-\tfrac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] dy$$

$$\cdot \int_{-\infty}^{y} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\,\frac{y-\mu_2}{\sigma_2}\right]^2\right\} dx,$$

and ν_{i2} is obtained from ν_{i1} by interchanges of x and y and of the subscripts 1 and 2. Calculation of the inner integral followed by the substitution $y = \mu_2 + \sigma_2 z$ gives

$$\nu_{i1} = \int_{-\infty}^{\infty} (\mu_2 + \sigma_2 z)^i \varphi(z)\, \Phi\left[\frac{(\sigma_2 - \sigma_1\rho)\, z + \mu_2 - \mu_1}{\sigma_1(1-\rho^2)^{1/2}}\right] dz.$$

Let $\nu_{i1}(x)$ be ν_{i1} with μ_1 replaced by x. The derivative of this function with respect to x will be denoted by a prime. The calculation of $\nu_{i1}'(\mu_1)$, followed by the substitution $z = [\sigma_1(1-\rho^2)^{1/2}/a]\, u - (\mu_2 - \mu_1)(\sigma_2 - \sigma_1\rho)/a^2$, followed by the substitution $\mu_1 = \mu_2 + am$ gives

$$\nu_{i1}'(m) = -\varphi(m) \int_{-\infty}^{\infty} \left[\mu_2 + \frac{\sigma_2(\sigma_2 - \sigma_1\rho)}{a}\, m + \frac{\sigma_1\sigma_2(1-\rho^2)^{1/2}}{a}\, u\right]^i \varphi(u)\, du.$$

One can observe that $\nu_{i1}(\infty) = 0$. Hence

$$\nu_{i1}(m) = -\int_m^{\infty} \nu_{i1}'(n)\, dn.$$

TABLE III

STATISTICS RELATED TO $\max(\xi, \eta)$ AND $r[\tau, \max(\xi, \eta)]$

μ	σ	ρ	ν_1	$\mu_2^{1/2}$	γ_1	γ_2	A	C	B
−4.5	1.0	−1.0	0.0085	0.9807	0.116	−0.84		0.9947	
−4.0	0.8	−1.0	0.0083	0.9809	0.116	−0.72		0.9954	
	1.0	−1.0	0.0170	0.9653	0.186	−1.28		0.9888	
		−0.5	0.0062	0.9876	0.066	−0.12	1.0020		0.0106
−3.5	0.6	−1.0	0.0081	0.9810	0.117	−0.61		0.9959	
	0.8	−1.0	0.0177	0.9635	0.196	−1.13		0.9894	
		−0.5	0.0068	0.9857	0.078	−0.14	1.0018		0.0102
	1.0	−1.0	0.0323	0.9411	0.282	−1.86		0.9774	
		−0.5	0.0139	0.9753	0.113	−0.15	1.0032		0.0222
−3.0	0.4	−1.0	0.0080	0.9810	0.118	−0.50		0.9965	
	0.6	−1.0	0.0189	0.9610	0.210	−0.97		0.9899	
		−0.5	0.0080	0.9826	0.098	−0.18	1.0014		0.0098
	0.8	−1.0	0.0357	0.9351	0.310	−1.65		0.9774	
		−0.5	0.0164	0.9699	0.142	−0.19	1.0028		0.0226
	1.0	−1.0	0.0586	0.9059	0.403	−2.59		0.9563	
		−0.5	0.0293	0.9546	0.179	−0.17	1.0040		0.0436
		0.0	0.0086	0.9869	0.050	−0.06	0.9961		0.0172
−2.5	0.2	−1.0	0.0081	0.9807	0.120	−0.38		0.9969	
	0.4	−1.0	0.0207	0.9574	0.228	−0.81		0.9903	
		−0.5	0.0106	0.9770	0.131	−0.23	1.0004		0.0092
	0.6	−1.0	0.0406	0.9268	0.346	−1.43		0.9769	
		−0.5	0.0207	0.9613	0.186	−0.24	1.0018		0.0232
		0.0	0.0067	0.9864	0.072	−0.12	0.9975		0.0097
	0.8	−1.0	0.0676	0.8926	0.457	−2.30		0.9541	
		−0.5	0.0363	0.9425	0.232	−0.20	1.0030		0.0464
		0.0	0.0123	0.9797	0.085	−0.10	0.9947		0.0208
	1.0	−1.0	0.1012	0.8584	0.547	−3.52		0.9188	
		−0.5	0.0577	0.9233	0.258	−0.13	1.0024		0.0806
		0.0	0.0219	0.9720	0.088	−0.07	0.9891		0.0397
−2.0	0.0	0.0	0.0085	0.9799	0.125	−0.25	0.9973		0.0000
	0.2	−1.0	0.0238	0.9518	0.255	−0.61		0.9904	
		−0.5	0.0161	0.9658	0.189	−0.30	0.9978		0.0076
		0.0	0.0096	0.9781	0.130	−0.24	0.9969		0.0051
	0.4	−1.0	0.0482	0.9149	0.397	−1.15		0.9759	
		−0.5	0.0289	0.9462	0.259	−0.29	0.9990		0.0232
		0.0	0.0133	0.9729	0.142	−0.22	0.9953		0.0130
	0.6	−1.0	0.0809	0.8740	0.530	−1.94		0.9507	
		−0.5	0.0482	0.9232	0.315	−0.23	1.0002		0.0498
		0.0	0.0206	0.9648	0.153	−0.17	0.9917		0.0268
	0.8	−1.0	0.1208	0.8341	0.635	−3.08		0.9113	
		−0.5	0.0741	0.9001	0.343	−0.13	0.9996		0.0890
		0.0	0.0326	0.9553	0.152	−0.11	0.9849		0.0496
	1.0	−1.0	0.1666	0.7994	0.702	−4.66		0.8540	
		−0.5	0.1066	0.8806	0.338	−0.04	0.9946		0.1410
		0.0	0.0503	0.9471	0.133	−0.05	0.9728		0.0830
		0.5	0.0085	0.9914	0.021	−0.02	0.9856		0.0230

TABLE III—*Continued*

μ	σ	ρ	ν_1	$\mu_2^{1/2}$	γ_1	γ_2	A	C	B
−1.5	0.0	0.0	0.0293	0.9425	0.295	−0.36	0.9901		0.0000
	0.2	−1.0	0.0607	0.8966	0.471	−0.78		0.9740	
		−0.5	0.0458	0.9185	0.381	−0.33	0.9918		0.0194
		0.0	0.0319	0.9398	0.296	−0.33	0.9889		0.0150
		0.5	0.0195	0.9600	0.214	−0.31	0.9888		0.0106
		1.0	0.0094	0.9780	0.136	−0.11		0.9976	
	0.4	−1.0	0.1016	0.8473	0.632	−1.44		0.9456	
		−0.5	0.0699	0.8909	0.448	−0.22	0.9936		0.0516
		0.0	0.0401	0.9324	0.292	−0.25	0.9847		0.0351
		0.5	0.0152	0.9702	0.152	−0.22	0.9868		0.0176
	0.6	−1.0	0.1499	0.8007	0.757	−2.48		0.9007	
		−0.5	0.1016	0.8639	0.476	−0.07	0.9932		0.0986
		0.0	0.0547	0.9228	0.273	−0.14	0.9762		0.0645
		0.5	0.0152	0.9745	0.108	−0.13	0.9824		0.0262
	0.8	−1.0	0.2039	0.7613	0.828	−3.97		0.8351	
		−0.5	0.1403	0.8422	0.455	0.05	0.9874		0.1600
		0.0	0.0762	0.9146	0.228	−0.05	0.9614		0.1056
		0.5	0.0195	0.9757	0.074	−0.06	0.9728		0.0416
	1.0	−1.0	0.2623	0.7333	0.847	−5.85		0.7456	
		−0.5	0.1851	0.8296	0.398	0.10	0.9726		0.2330
		0.0	0.1048	0.9120	0.167	0.00	0.9381		0.1584
		0.5	0.0293	0.9773	0.040	−0.01	0.9548		0.0684
−1.0	0.0	0.0	0.0833	0.8667	0.582	−0.24	0.9708		0.0000
	0.2	−1.0	0.1360	0.8070	0.784	−0.69		0.9383	
		−0.5	0.1122	0.8354	0.669	−0.08	0.9760		0.0442
		0.0	0.0882	0.8644	0.564	−0.19	0.9678		0.0378
		0.5	0.0640	0.8943	0.464	−0.28	0.9644		0.0308
		1.0	0.0405	0.9255	0.368	−0.03		0.9892	
	0.4	−1.0	0.1952	0.7531	0.935	−1.51		0.8863	
		−0.5	0.1500	0.8061	0.694	0.12	0.9780		0.1050
		0.0	0.1026	0.8594	0.507	−0.07	0.9582		0.0822
		0.5	0.0545	0.9149	0.342	−0.21	0.9556		0.0550
		1.0	0.0119	0.9736	0.181	0.04		0.9977	
	0.6	−1.0	0.2591	0.7096	1.010	−2.85		0.8095	
		−0.5	0.1952	0.7840	0.643	0.27	0.9726		0.1818
		0.0	0.1265	0.8557	0.408	0.04	0.9400		0.1371
		0.5	0.0545	0.9285	0.230	−0.10	0.9416		0.0812
	0.8	−1.0	0.3261	0.6807	1.004	−4.60		0.7042	
		−0.5	0.2467	0.7736	0.534	0.30	0.9552		0.2700
		0.0	0.1592	0.8586	0.284	0.08	0.9114		0.2026
		0.5	0.0640	0.9393	0.129	−0.02	0.9180		0.1172
	1.0	−1.0	0.3956	0.6693	0.950	−6.31		0.5722	
		−0.5	0.3031	0.7779	0.416	0.21	0.9232		0.3624
		0.0	0.1996	0.8721	0.174	0.05	0.8718		0.2749
		0.5	0.0833	0.9538	0.052	0.00	0.8820		0.1664
−0.5	0.0	0.0	0.1978	0.7439	1.014	0.46	0.9295		0.0000
	0.2	−1.0	0.2697	0.6837	1.210	0.11		0.8685	
		−0.5	0.2383	0.7144	1.063	0.72	0.9424		0.0914
		0.0	0.2048	0.7458	0.943	0.47	0.9226		0.0837
		0.5	0.1687	0.7786	0.842	0.24	0.9084		0.0752
		1.0	0.1295	0.8143	0.774	0.48		0.9667	

TABLE III—*Continued*

STATISTICS RELATED TO $\max(\xi, \eta)$ AND $r[\tau, \max(\xi, \eta)]$

μ	σ	ρ	ν_1	$\mu_2^{1/2}$	γ_1	γ_2	A	C	B
−0.5	0.4	−1.0	0.3438	0.6381	1.291	−0.78		0.7763	
		−0.5	0.2877	0.6957	0.964	0.86	0.9422		0.1980
		0.0	0.2252	0.7529	0.750	0.46	0.9015		0.1707
		0.5	0.1535	0.8136	0.603	0.14	0.8812		0.1392
		1.0	0.0680	0.8896	0.578	0.35		0.9876	
	0.6	−1.0	0.4192	0.6109	1.237	−2.21		0.6487	
		−0.5	0.3438	0.6923	0.756	0.74	0.9238		0.3124
		0.0	0.2574	0.7690	0.501	0.36	0.8660		0.2607
		0.5	0.1535	0.8476	0.355	0.10	0.8458		0.2004
		1.0	0.0202	0.9601	0.380	0.20		0.9975	
	0.8	−1.0	0.4956	0.6049	1.102	−3.66		0.4909	
		−0.5	0.4048	0.7063	0.534	0.47	0.8856		0.4242
		0.0	0.2993	0.7971	0.280	0.19	0.8178		0.3494
		0.5	0.1687	0.8842	0.153	0.06	0.8000		0.2648
	1.0	−1.0	0.5727	0.6210	0.991	−4.35		0.3179	
		−0.5	0.4696	0.7380	0.393	0.25	0.8314		0.5236
		0.0	0.3491	0.8388	0.153	0.07	0.7608		0.4314
		0.5	0.1978	0.9284	0.044	0.01	0.7448		0.3324
0.0	0.0	0.0	0.3989	0.5838	1.641	2.41	0.8564		0.0000
	0.2	−1.0	0.4787	0.5393	1.766	3.15		0.7417	
		−0.5	0.4442	0.5680	1.556	2.52	0.8802		0.1760
		0.0	0.4068	0.5954	1.418	2.06	0.8398		0.1680
		0.5	0.3656	0.6215	1.340	1.72	0.8044		0.1608
		1.0	0.3192	0.6466	1.374	1.36		0.9279	
	0.4	−1.0	0.5585	0.5177	1.625	3.06		0.5794	
		−0.5	0.4983	0.5759	1.163	1.91	0.8682		0.3472
		0.0	0.4297	0.6288	0.925	1.29	0.7952		0.3181
		0.5	0.3478	0.6775	0.837	0.93	0.7380		0.2952
		1.0	0.2394	0.7230	1.217	0.35		0.9682	
	0.6	−1.0	0.6383	0.5221	1.323	2.14		0.3831	
		−0.5	0.5585	0.6067	0.727	1.03	0.8242		0.4944
		0.0	0.4652	0.6808	0.470	0.58	0.7344		0.4406
		0.5	0.3478	0.7477	0.380	0.37	0.6688		0.4012
		1.0	0.1596	0.8090	1.233	−0.25		0.9888	
	0.8	−1.0	0.7181	0.5517	1.075	1.19		0.1813	
		−0.5	0.6232	0.6570	0.454	0.42	0.7610		0.6088
		0.0	0.5109	0.7477	0.209	0.17	0.6688		0.5350
		0.5	0.3656	0.8284	0.111	0.09	0.6036		0.4828
		1.0	0.0798	0.9020	1.378	−0.34		0.9978	
	1.0	−1.0	0.7979	0.6028	0.995	0.87		0.0000	
		−0.5	0.6910	0.7229	0.375	0.24	0.6917		0.6917
		0.0	0.5642	0.8256	0.137	0.06	0.6056		0.6056
		0.5	0.3989	0.9170	0.035	0.01	0.5453		0.5453
0.5	0.0	0.0	0.6978	0.4129	2.583	7.25	0.7472		0.0000
	0.2	−1.0	0.7697	0.3967	2.407	15.13		0.5197	
		−0.5	0.7383	0.4215	2.068	5.71	0.7750		0.3194
		0.0	0.7048	0.4418	1.897	4.93	0.7062		0.3115
		0.5	0.6687	0.4562	1.870	4.63	0.6416		0.3100
		1.0	0.6295	0.4624	2.122	−0.39		0.8927	
	0.4	−1.0	0.8438	0.4156	1.712	16.69		0.2519	
		−0.5	0.7877	0.4720	1.104	2.54	0.7298		0.5556
		0.0	0.7252	0.5163	0.860	1.83	0.6222		0.5259
		0.5	0.6535	0.5455	0.837	1.64	0.5190		0.5256
		1.0	0.5680	0.5398	1.613	−4.13		0.9660	

TABLE III—*Continued*

μ	σ	ρ	ν_1	$\mu_2^{1/2}$	γ_1	γ_2	A	C	B
0.5	0.6	−1.0	0.9192	0.4649	1.167	12.89		0.0080	
		−0.5	0.8438	0.5483	0.539	0.78	0.6574		0.6998
		0.0	0.7574	0.6155	0.304	0.45	0.5427		0.6491
		0.5	0.6535	0.6640	0.246	0.36	0.4264		0.6478
		1.0	0.5202	0.6458	0.986	−3.34		0.9945	
	0.8	−1.0	0.9956	0.5359	0.975	8.74		−0.1809	
		−0.5	0.9048	0.6391	0.367	0.25	0.5858		0.7830
		0.0	0.7993	0.7253	0.136	0.08	0.4800		0.7190
		0.5	0.6687	0.7953	0.047	0.04	0.3680		0.7114
	1.0	−1.0	1.0727	0.6210	0.991	6.05		−0.3179	
		−0.5	0.9696	0.7380	0.393	0.25	0.5236		0.8314
		0.0	0.8491	0.8388	0.153	0.07	0.4314		0.7608
		0.5	0.6978	0.9284	0.044	0.01	0.3324		0.7448
1.0	0.0	0.0	1.0833	0.2615	4.115	19.94	0.6066		0.0000
	0.2	−1.0	1.1360	0.2825	2.763	59.97		0.1515	
		−0.5	1.1122	0.3039	2.225	9.04	0.6074		0.5366
		0.0	1.0882	0.3177	2.020	7.90	0.5143		0.5266
		0.5	1.0640	0.3225	2.058	7.94	0.4268		0.5348
		1.0	1.0405	0.3151	2.544	−17.15		0.9028	
	0.4	−1.0	1.1952	0.3552	1.319	41.63		−0.1899	
		−0.5	1.1500	0.4066	0.716	1.77	0.5206		0.7754
		0.0	1.1026	0.4418	0.510	1.32	0.3997		0.7456
		0.5	1.0545	0.4562	0.504	1.29	0.2754		0.7666
		1.0	1.0119	0.4337	0.901	−8.81		0.9884	
	0.6	−1.0	1.2591	0.4517	0.903	22.35		−0.3861	
		−0.5	1.1952	0.5279	0.344	0.30	0.4500		0.8666
		0.0	1.1265	0.5854	0.140	0.17	0.3341		0.8245
		0.5	1.0545	0.6189	0.088	0.14	0.2030		0.8476
	0.8	−1.0	1.3261	0.5582	0.878	12.47		−0.5004	
		−0.5	1.2467	0.6530	0.332	0.13	0.3998		0.9052
		0.0	1.1592	0.7306	0.109	0.02	0.2976		0.8569
		0.5	1.0640	0.7883	0.017	0.01	0.1746		0.8752
	1.0	−1.0	1.3956	0.6693	0.950	7.72		−0.5722	
		−0.5	1.3031	0.7779	0.416	0.21	0.3624		0.9232
		0.0	1.1996	0.8721	0.174	0.05	0.2749		0.8718
		0.5	1.0833	0.9538	0.052	0.00	0.1664		0.8820
1.5	0.0	0.0	1.5293	0.1483	6.865	57.58	0.4505		0.0000
	0.2	−1.0	1.5607	0.2161	2.048	146.92		−0.3389	
		−0.5	1.5458	0.2336	1.505	7.58	0.3810		0.7800
		0.0	1.5319	0.2427	1.338	6.69	0.2911		0.7657
		0.5	1.5195	0.2432	1.377	6.91	0.2092		0.7806
		1.0	1.5094	0.2344	1.674	−40.84		0.9570	
	0.4	−1.0	1.6016	0.3413	0.788	50.16		−0.5895	
		−0.5	1.5699	0.3830	0.329	0.56	0.3000		0.9244
		0.0	1.5401	0.4086	0.181	0.46	0.2003		0.8988
		0.5	1.5152	0.4158	0.168	0.45	0.1026		0.9210
	0.6	−1.0	1.6499	0.4734	0.700	21.17		−0.6785	
		−0.5	1.6016	0.5367	0.244	0.04	0.2646		0.9592
		0.0	1.5547	0.5818	0.065	0.03	0.1705		0.9290
		0.5	1.5152	0.6036	0.018	0.03	0.0706		0.9516
	0.8	−1.0	1.7039	0.6044	0.765	10.99		−0.7210	
		−0.5	1.6403	0.6859	0.308	0.04	0.2456		0.9698
		0.0	1.5762	0.7506	0.098	−0.01	0.1609		0.9371
		0.5	1.5195	0.7929	0.010	0.00	0.0642		0.9576

TABLE III—*Continued*

STATISTICS RELATED TO $\max(\xi, \eta)$ AND $r[\tau, \max(\xi, \eta)]_s$

μ	σ	ρ	ν_1	$\mu_2^{1/2}$	γ_1	γ_2	A	C	B
1.5	1.0	−1.0	1.7623	0.7333	0.847	6.65		−0.7456	
		−0.5	1.6851	0.8296	0.398	0.10	0.2330		0.9726
		0.0	1.6048	0.9120	0.167	0.00	0.1584		0.9381
		0.5	1.5293	0.9773	0.040	−0.01	0.0684		0.9548
2.0	0.0	0.0	2.0085	0.0755	12.323	189.16	0.3014		0.0000
	0.2	−1.0	2.0238	0.1942	0.855	155.25		−0.7344	
		−0.5	2.0161	0.2060	0.543	2.66	0.1760		0.9358
		0.0	2.0096	0.2113	0.463	2.46	0.1180		0.9231
	0.4	−1.0	2.0482	0.3544	0.474	36.11		−0.8262	
		−0.5	2.0289	0.3837	0.152	0.05	0.1424		0.9856
		0.0	2.0133	0.3998	0.046	0.09	0.0792		0.9689
	0.6	−1.0	2.0809	0.5091	0.535	14.76		−0.8465	
		−0.5	2.0482	0.5570	0.185	−0.06	0.1374		0.9946
		0.0	2.0206	0.5883	0.038	−0.01	0.0734		0.9759
	0.8	−1.0	2.1208	0.6571	0.621	7.80		−0.8525	
		−0.5	2.0741	0.7227	0.259	−0.06	0.1386		0.9960
		0.0	2.0326	0.7714	0.077	−0.03	0.0767		0.9756
	1.0	−1.0	2.1666	0.7994	0.702	4.82		−0.8540	
		−0.5	2.1066	0.8806	0.338	−0.04	0.1410		0.9946
		0.0	2.0503	0.9471	0.133	−0.05	0.0830		0.9728
		0.5	2.0085	0.9914	0.021	−0.02	0.0230		0.9856
2.5	0.2	−1.0	2.5081	0.1936	0.292	88.51		−0.9177	
	0.4	−1.0	2.5207	0.3727	0.293	20.56		−0.9339	
		−0.5	2.5106	0.3905	0.080	−0.05	0.0580		1.0010
	0.6	−1.0	2.5406	0.5428	0.381	9.01		−0.9312	
		−0.5	2.5207	0.5758	0.128	−0.10	0.0644		1.0034
		0.0	2.5067	0.5946	0.021	−0.02	0.0270		0.9930
	0.8	−1.0	2.5676	0.7043	0.468	5.02		−0.9252	
		−0.5	2.5363	0.7535	0.190	−0.12	0.0726		1.0036
		0.0	2.5124	0.7862	0.049	−0.04	0.0324		0.9916
	1.0	−1.0	2.6012	0.8584	0.547	3.21		−0.9188	
		−0.5	2.5577	0.9233	0.258	−0.13	0.0806		1.0024
		0.0	2.5219	0.9720	0.088	−0.07	0.0397		0.9891
3.0	0.4	−1.0	3.0080	0.3864	0.167	10.48		−0.9770	
	0.6	−1.0	3.0189	0.5679	0.247	5.14		−0.9709	
		−0.5	3.0080	0.5883	0.076	−0.09	0.0274		1.0036
	0.8	−1.0	3.0357	0.7409	0.327	3.06		−0.9637	
		−0.5	3.0164	0.7749	0.124	−0.13	0.0354		1.0040
	1.0	−1.0	3.0586	0.9059	0.403	2.02		−0.9563	
		−0.5	3.0293	0.9546	0.179	−0.17	0.0436		1.0040
		0.0	3.0086	0.9869	0.050	−0.06	0.0172		0.9961
3.5	0.6	−1.0	3.5081	0.5838	0.145	2.76		−0.9885	
	0.8	−1.0	3.5177	0.7662	0.213	1.78		−0.9833	
		−0.5	3.5068	0.7878	0.071	−0.11	0.0158		1.0028
	1.0	−1.0	3.5323	0.9411	0.282	1.22		−0.9774	
		−0.5	3.5139	0.9753	0.113	−0.15	0.0222		1.0032
4.0	0.8	−1.0	4.0083	0.7821	0.128	0.99		−0.9927	
	1.0	−1.0	4.0170	0.9653	0.186	0.71		−0.9888	
		0.5	4.0062	0.9876	0.066	−0.12	0.0106		1.0020
4.5	1.0	−1.0	4.5085	0.9807	0.116	0.39		−0.9947	

A separate calculation for each $i=1, \ldots, 4$ produces, with, again, $\alpha=(\mu_1-\mu_2)/a$,

$$\nu_{11}=\mu_2\,\Phi(-\alpha)+[\sigma_2\,(\sigma_2-\sigma_1\,\rho)/a]\,\varphi(\alpha),$$

$$\nu_{21}=(\mu_2^2+\sigma_2^2)\,\Phi(-\alpha)+[2\,\mu_2\,\sigma_2\,(\sigma_2-\sigma_1\,\rho)/a+(\mu_1-\mu_2)\,\sigma_2^2\,(\sigma_2-\sigma_1\,\rho)^2/a^3]\,\varphi(\alpha),$$

$$\nu_{31}=(\mu_2^3+3\,\mu_2\,\sigma_2^2)\,\Phi(-\alpha)+[3\,(\mu_2^2+\sigma_2^2)\,\sigma_2\,(\sigma_2-\sigma_1\,\rho)/a-\sigma_2^3\,(\sigma_2-\sigma_1\,\rho)^3/a^3$$
$$+3\mu_2\,(\mu_1-\mu_2)\,\sigma_2^2\,(\sigma_2-\sigma_1\,\rho)^2/a^3+(\mu_1-\mu_2)^2\,\sigma_2^3\,(\sigma_2-\sigma_1\,\rho)^3/a^5]\,\varphi(\alpha),$$

$$\nu_{41}=(\mu_2^4+6\,\mu_2^2\,\sigma_2^2+3\,\sigma_2^4)\,\Phi(-\alpha)+[(4\,\mu_2^3+12\,\mu_2\,\sigma_2^2)\,\sigma_2\,(\sigma_2-\sigma_1\,\rho)/a$$
$$-4\,\mu_2\,\sigma_2^3\,(\sigma_2-\sigma_1\,\rho)^3/a^3+6\,(\mu_2^2+\sigma_2^2)(\mu_1-\mu_2)\,\sigma_2^2\,(\sigma_2-\sigma_1\,\rho)^2/a^3$$
$$-3\,(\mu_1-\mu_2)\,\sigma_2^4\,(\sigma_2-\sigma_1\rho)^4/a^5+4\,\mu_2\,(\mu_1-\mu_2)^2\,\sigma_2^3\,(\sigma_2-\sigma_1\,\rho)^3/a^5$$
$$+(\mu_1-\mu_2)^3\,\sigma_2^4\,(\sigma_2-\sigma_1\,\rho)^4/a^7]\,\varphi(\alpha).$$

In obtaining these results, one makes use of the identities $a^2=\sigma_1^2+\sigma_2^2-2\,\sigma_1\,\sigma_2\,\rho=(\sigma_2-\sigma_1\,\rho)^2+\sigma_1^2\,(1-\rho^2)=(\sigma_1-\sigma_2\,\rho)^2+\sigma_2^2\,(1-\rho^2)=\sigma_1\,(\sigma_1-\sigma_2\,\rho)+\sigma_2\,(\sigma_2-\sigma_1\,\rho)$. As observed above, ν_{i2} can be written immediately from ν_{i1}, and the sum $\nu_i=\nu_{i1}+\nu_{i2}$ is available. These sums can be reduced respectively to (2) through (5).

It remains to calculate the coefficient of correlation (6). We assume that the expected value and variance of τ are 0 and 1, respectively. This involves no loss of generality because a linear transformation of τ will not alter any of the correlations among the random variables considered. The probability density of ξ, η, and τ is

$$\varphi(x,\,y,\,t)=(2\pi)^{-3/2}\,(\sigma_1\,\sigma_2)^{-1}\,R^{-1/2}\exp(-\tfrac{1}{2}\,Q),$$

where

$$R=1+2\rho\rho_1\,\rho_2-\rho^2-\rho_1^2-\rho_2^2,$$

$$Q=|R|^{-1}\,[(1-\rho_2^2)\,X^2+(1-\rho_1^2)\,Y^2+(1-\rho^2)\,t^2+2\,(\rho_1\,\rho_2-\rho)\,XY$$
$$+2\,(\rho\rho_2-\rho_1)\,Xt+2\,(\rho\rho_1-\rho_2)\,Yt],$$

with $$x=\mu_1+\sigma_1\,X, \qquad y=\mu_2+\sigma_2\,Y.$$

The expected value of the product $\tau\max(\xi,\eta)$ is

$$\mathrm{E}[\tau\max(\xi,\,\eta)]=(2\pi)^{-3/2}\,(\sigma_1\,\sigma_2)^{-1}\,R^{-1/2}$$
$$\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max(x,\,y)\,\exp\left[\frac{-1}{2\,(1-\rho^2)}\,(X^2+Y^2-2\rho XY)\right]dx\,dy$$
$$\cdot\int_{-\infty}^{\infty}t\,\exp\left\{-\frac{1-\rho^2}{2R}\left[t+\frac{\rho\rho_2-\rho_1}{1-\rho^2}\,X+\frac{\rho\rho_1-\rho_2}{1-\rho^2}\,Y\right]^2\right\}dt,$$

which reduces to $\mathrm{E}_1+\mathrm{E}_2$ with

$$\mathrm{E}_1=(2\pi)^{-1}\,(1-\rho^2)^{-3/2}\int_{-\infty}^{\infty}(\mu_1+\sigma_1\,X)\,dX$$
$$\cdot\int_{-\infty}^{(\mu_1-\mu_2+\sigma_1 X)/\sigma_2}[(\rho_1-\rho\rho_2)\,X+(\rho_2-\rho\rho_1)\,Y]\,\exp\left[\frac{-1}{2(1-\rho^2)}\,(X^2+Y^2-2\rho XY)\right]dy,$$

and E_2 obtained from E_1 by an interchange of X and Y and of the subscripts 1 and 2.

From this point the calculation, although not brief, involves manipulations similar to those used to obtain the ν_i. One obtains the surprisingly simple result

$$E[\tau \max(\xi, \eta)] = \sigma_1 \rho_1 \Phi(\alpha) + \sigma_2 \rho_2 \Phi(-\alpha).$$

Since $E(\tau)=0$ and $V(\tau)=1$, we divide by the standard deviation of $\max(\xi, \eta)$ to obtain (6).

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