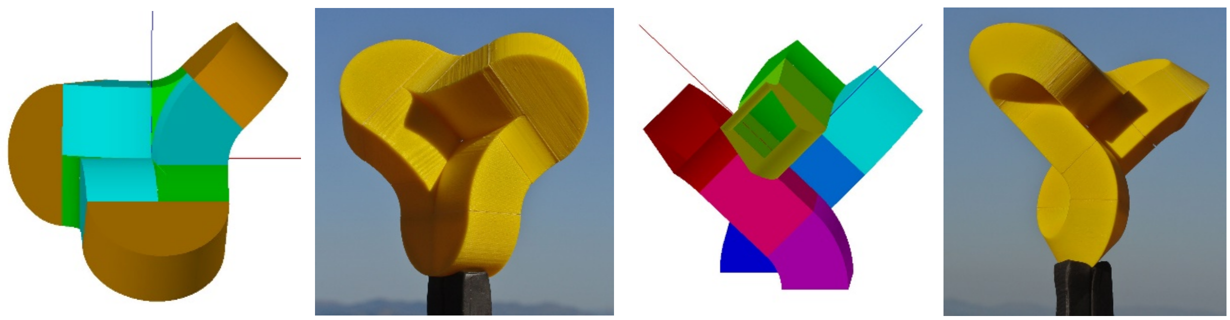
Tetra-Tangle of Four Bow-Tie Links

Carlo H. Séquin
EECS Computer Science Division, U.C. Berkeley
sequin@cs.berkeley.edu

Figure 1: "Tetra-Tangle of Four Bow-Tie Links" by Carlo H. Séquin, 9in × 9in × 9in.

This Bow-Tie Tetra-Tangle was constructed specifically for the Exhibition of Mathematical Art to be held at the Joint Mathematics Meeting (JMM) in San Antonio, TX, in January 2015. It resulted from an extension of the work on "LEGO® Knots" reported at Bridges Seoul Conference in August 2014. In that project, a small set of snap-together, tubular parts had been developed to permit the construction of sculptural maquettes in the form of prismatic extrusions along modular space curves composed of circular arcs. The concept of such modular components was inspired by Henk van Putten's "Borsalino" sculpture [3] presented at Bridges 2013 in Enschede. It is composed of two types of modules (Fig.2a): the three (brown) end-caps that form tight 180° turns, and the six (green and cyan) connector pieces, which exhibit gentler bends through an angle of 45°. Henk van Putten's sculptures, while modular in their geometry, are constructed as coherent solid objects. But I thought it would be a lot of fun, if one could play with such geometrical building blocks in real time in a tangible manner. So I constructed some LEGO®-like, snap-together parts with inexpensive layered-manufacturing techniques. For most of my experiments I used Fused Deposition Modeling (FDM) machines from Stratasys. Figure 2b shows a realization of the "Borsalino" shape from nine plastic parts built on such a machine.

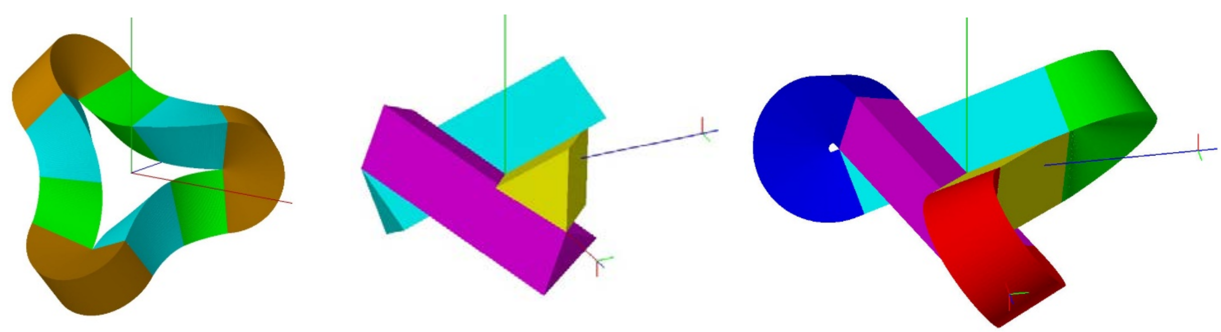


(a) (b) (c) (d)

Figure 2: Borsalinos: (a) 9-part geometry by Henk van Putten; (b) realization with FDM; (c) extension pieces inserted between curved connectors; (d) resulting "Bow-Tie Borsalino".

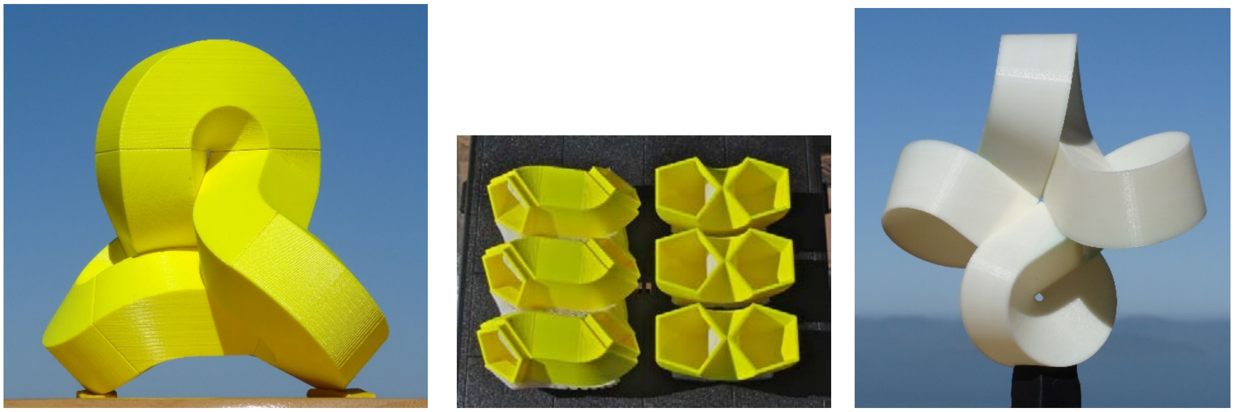
Playing with the initial set of parts was indeed very inspiring, and I soon added additional modules into my building-block set: curved connector pieces and end-caps with different bending radii. During these experiments, it occurred to me that stretching the connection in the middle of each curved connector pair (Fig.2c) would lead to another interesting configuration. When the inserted tube is of just the right length, the two square end-cross-sections that previously were connected by an end-cap shift past one another until they are located corner to corner (Fig.2c) Now, to close off the two open ends, we need a new end-cap that sweeps the square cross section through a half-circle around one vertex, parallel to one of its face diagonals. This is the same as sweeping a "rhombic" cross section, i.e., a square with an azimuthal rotation of 45°, along an arc with a radius enlarged by $\sqrt{2}$. This rhombic end-cap together with two attached curved connector pieces forms an interesting sculptural "Bow-Tie" shape. Figure 3c shows the resulting "Bow-Tie Borsalino."

In a different set of experiments with Michelle Galemmo we investigated what happens when the square cross-section in these "Borsalino" shapes is replaced with an equilateral triangle [1]. In addition to some "Borsalino" shapes with twisted arms (Fig.3a), a variety of different "Bow-Tie Loops" can be formed. We start by placing three prismatic prisms flush against one another (Fig.3b) and then truncate their length to the point where their outer edges intersect. Wherever a pair of end-faces of these prisms share a vertex, we add a specially constructed, "helically" twisting end-cap. The result is a single, closed sweep of the given triangular cross section, forming a 3-lobe Bow-Tie loop (Fig.3c).



(a) (b) (c)

Figure 3: (a) Borsalino shape with triangular cross-section; (b) a flush tangle of three prismatic beams cut to proper length; (c) Bow-Tie-Loop formed with three helically twisted end-caps;

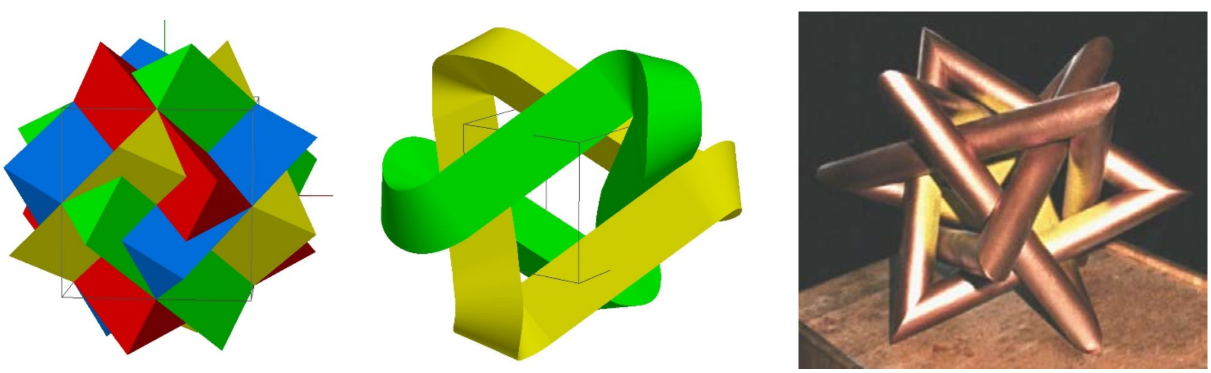


(a) (b) (c)

Figure 4: More Bow-Tie Loops: (a) three lobes with a pentagonal beam, (b) the six parts used in its construction; (c) four-lobe Bow-Tie Loop with a triangular cross section.

There are more ways to make such Bow-Tie Loops. Figures 2d and 3c show ways to make 3-lobe Bow-Tie Loops with beams that have regular quadrilaterals and triangular cross sections, respectively. Figure 4a demonstrates that this can also be done with a beam with a pentagonal cross-section. Figure 4b shows the six plastic parts that were used in its construction. Alternatively, one can pack up to five triangular prisms tightly and symmetrically around the origin so that each beam is in flush face-to-face contact with its nearest two neighbors. Again the beams are truncated to the point where their outer edges intersect, and neighbors are then joined with custom-made helical end-caps. Figure 4c shows a 4-lobe realization.

In the implementation of these various Bow-Tie Loops, I experimented with different ways of partitioning and modularizing the beam. For the shapes shown in Figures 3c and 4a, the beam was split into six parts. Figure 4c shows an assembly of only four parts, where each one of them is a Bow-Tie lobe starting and ending at the origin. It results in nice smooth lobes – but it was very difficult to assemble!



(a) (b) (c)

Figure 5: (a) Geometry of the TETRAXIS® puzzle; (b) virtual assembly of two of the four loops; (c) "Tetra-Tangle" made from 4"-diameter tubes (1983).

With the sculpture "**Tetra-Tangle of Four Bow-Tie Links**" (Fig.1), constructed for JMM 2015, I tackled the problem how several such Bow-Tie Loops might be mutually interlinked (Fig.5b). The key inspiration came from the TETRAXIS® puzzle, which consists of four sets of three mutually parallel, 3-sided prisms, pointing in four different tetrahedral directions (Fig.5a). There are eight points where three beams join in a flush, symmetric tangle, and six points with tight, 4-beam tangles. When two triangular prism-end-faces that share a common vertex are closed off with a connecting sweep, a loose "Bow-Tie" is formed. If all twelve pairs of adjoining triangular end-faces are connected in this way, the result is a link of four mutually interlocking, 3-sided Bow-Tie Loops. This represents an alternating 12-crossing link that has the same connectivity as the "Tetra-Tangle," which I constructed from 4"-diameter card-board tubes in 1983 (Fig.5c). The new geometry has been realized as four differently colored loops, each composed of six tubular snap-together parts (Fig.6) fabricated on an FDM machine. The overall assembly takes place one ring at a time. For the incorporation of the fourth loop, first the straight segments are introduced into the partial assembly (Fig.7a) and then the twisted end-caps are applied. The last end-cap to be applied took a little extra force, since its two insertion sleeves are not parallel.

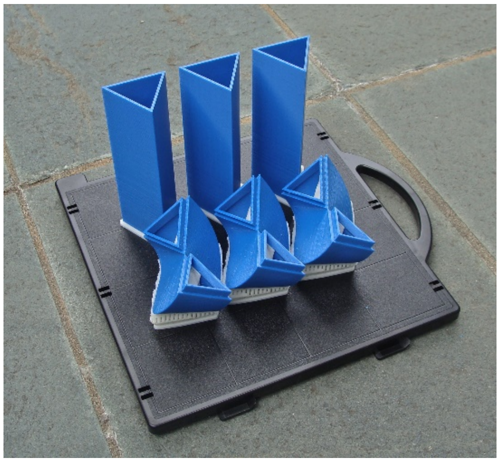
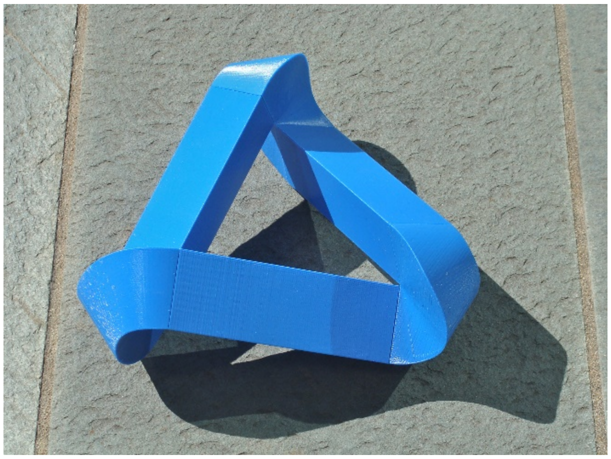
(a)  (b) 

Figure 6: Fabrication of the first Bow-Tie loop: (a) one batch of six parts fabricated with FDM; (b) assembly of those six parts into a closed loop.

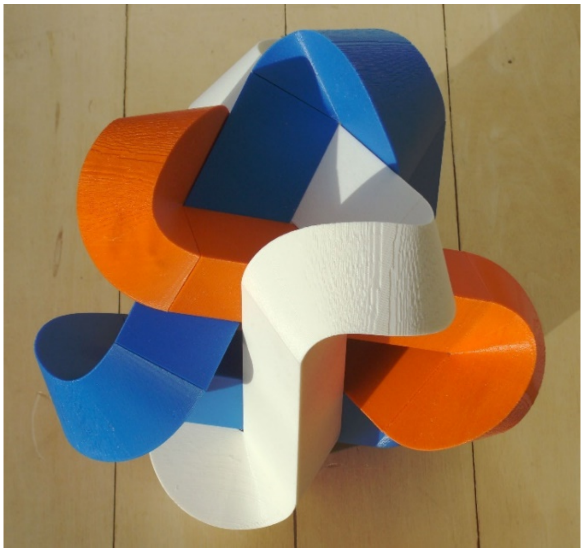
(a)  (b) 

Figure 7: Assembly of the Bow-Tie Tetra-Tangle: (a) 3 loops assembled; (b) all 4 loops assembled.

References:

[1] M. Galemmo and C. H. Séquin: *Tria Tubes.* Bridges Conf. Proc, Seoul, Korea, Aug. 14-19, 2014, 373-376, -- http://archive.bridgesmathart.org/2014/bridges2014-373.html

[2] C. H. Séquin and M. Galemmo: *LEGO-Knots.* Bridges Conf. Proc, Seoul, Korea, Aug. 14-19, 2014, 261-270, -- http://archive.bridgesmathart.org/2014/bridges2014-261.html

[3] H. van Putten: *Limitation or addition? The Borsalino shape.* Bridges 2013, Mathematical Art Gallery, -- http://gallery.bridgesmathart.org/exhibitions/2013-bridges-conference/henk-van-putten