CS184 Fall 2002 — Linear Algebra Review

Introduction: Some of these questions may appear difficult, and a review of linear algebra should help. Some relevant topics are dot products, cross products, triple products, and determinants. Many textbooks have useful formulas for the area of a triangle and volume of a pyramid.

Question 1

You are given two vectors, i and j, in 2D (also called *plane* vectors), that satisfy the following properties: $i \cdot i = j \cdot j = 1$ and $i \cdot j = 0$.

a. Is there a 2d vector k that is not equal to i so that $k \cdot k = 1$ and $k \cdot j = 0$? What is it? How many such vectors are there?

b. Is there a 2d vector k so that $k \cdot k = 1$ and $k \cdot j = 0$ a nd $k \cdot i = 0$?

c. How do the answers to **a.** and **b.** change if all the vectors are considered in 3D?

Question 2

For three points in 2D, (x_1, y_1), (x_2, y_2), and (x_3, y_3), show that the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

is proportional to the area of the triangle whose corners are the three points. What is the value of the determinant when the three points are colinear (lie on the same line)? How could you make this into a test to check whether three points are colinear? Would this be a good test?

Question 3

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Show that if (i, j, k) is equal to $(1, 2, 3)$ or $(2, 3, 1)$ or $(3, 1, 2)$ then $e_i \times e_j = e_k$. Then show that if (i, j, k) is equal to $(3, 2, 1)$ or $(2, 1, 3)$ or $(1, 3, 2)$ then $e_i \times e_j = -e_k$. (Note: This problem is really looking at left and right handed coordinate systems.)

Question 4

For four points in 3D, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4), show that the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

is proportional to the volume of the prism whose corners are the four points. What is the value of the determinant if the four points lie in a plane? How could you make this into a test to check whether the four points are copla nar (lie in the same plane)? Would this be a good test?

Question 5

The equation of a plane in 3D is $ax + by + cz + d = 0$. Given three points in space find a, b, c, and d so that the equation describes the plane containing the three points.

Question 6

Let p_1, p_2, p_3 be points in 3D. Consider the cross product $n = (p_2 - p_1) \times (p_3 - p_1)$. What does this vector mean geometrically? Let p be any point in the plane containing p_1, p_2, p_3. What is the geometric relationship between $p - p_1$ and n (look at their dot product)? Show that if $p = (x, y, z)$ that $n \cdot (p - p_1) = 0$ is an equation for the plane containing p_1, p_2, and p_3.

Question 7

Given a square matrix A, we say it is Orthogonal if and only if A's inverse exists and is equal to A's transpose. Show that for any value of θ the matrix $M(\theta)$, as defined below, is orthogonal.

$$M(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Also show that $M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2)$, and therefore that the inverse of $M(\theta)$ must be $M(-\theta)$ for all θ. Finally, for point $p = (1, 0)$, what is the locus of points defined by $M(\theta)p$ for all possible values of θ.

Question 8

You are given four vectors in 2D x_1, x_2, b_1, b_2, and told that there is a matrix M so that $Mx_1 = b_1$ and $Mx_2 = b_2$, but you do NOT have M. For an arbitrary x_3 when and how can you compute Mx_3. When and how can you compute M?

Question 9

Write the parametric equation for a line through points p_1 and p_2. Now write the parametric equation for a plane through points p_1, p_2, and p_3. Show how to restrict these parametric equations to specify only the points on the line segment between p_1 and p_2 for the line, and only the points inside the triangle formed by p_1, p_2, and p_3 for the plane.

Question 10

Find the length of the vector $(2, 4, -3)$. Use the dot product to find the angle between $(2, 4, -3)$ and $(1, 0, 0)$ express your answer in radians.